Improved Projection GMM-LM Tests for Linear Restrictions*

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Abstract

We extend the two-step GMM-LM projection test for subvectors in Chaudhuri and Zivot (2011) to testing null hypotheses on linear restrictions on a parameter vector. This extension retains all the asymptotic properties of the original test under more general scenarios. A key aspect of our paper is that the identification of the linear restrictions being tested may be rate-entangled because we allow for multiple rates (strength) of identification for the elements of the parameter vector. This leads to novel issues related to the fundamentally important idea of the benchmark for feasible local efficiency when we study the two-step projection test's local efficiency by establishing its asymptotic equivalence with a locally efficient infeasible plug-in test. This happens because, when the null hypothesis is false, the infeasible plug-in test is not invariant to the characterization of the restrictions *not* being tested, while the two-step projection test is invariant. We point out these issues and address them under a unified framework. We illustrate our general results with a simple example and a simulation study. Our results have a broader appeal since these same issues related to the benchmark for feasible local efficiency would also arise under our framework for any invariant test that seeks asymptotic equivalence with an infeasible but non-invariant test.

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1 Introduction

Consider a parameter vector $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$ whose unknown true value θ^0 satisfies the moment restrictions:

$$E[g(Z_t; \theta^0)] = 0 \tag{1}$$

where $\{Z_t\}_{t=1}^T$ are $\mathbb{R}^{d_z}$-valued random vectors, $g(.;\theta) : \mathbb{R}^{d_z} \times \Theta \mapsto \mathbb{R}^{d_g}$ is a known (up to θ) function, and $d_g \geq d_\theta$. Suppose that we are interested in testing:

$$\text{the null hypothesis } H_0 : R\theta^0 = r_0 \text{ against the alternative hypothesis } H : R\theta^0 \neq r_0 \tag{2}$$

where R is a fixed, full row-rank, $d_R \times d_\theta$ known matrix, and r_0 is a $d_R \times 1$ known vector, and $d_R < d_\theta$.

This paper extends the improved two-step GMM-LM projection subvector (of θ) test presented in Chaudhuri and Zivot (2011) and built on the original work of Robins (2004), to testing H_0 in (2). (Also see Chaudhuri (2008), Zivot and Chaudhuri (2009), Chaudhuri et al. (2010), Chaudhuri and Renault (2011).) For brevity, in the sequel we generally refer to this test and its extension as the *two-step test*.

Chaudhuri and Zivot (2011) established the following results in the context of testing for subvectors. First, allowing for identification failure of θ^0 in (1) as in Stock and Wright (2000), this test's asymptotic rejection probability of the true null can be non-trivially bounded from above, a property that the subvector-plug-in tests cannot possess in general [see Guggenberger et al. (2012a)]. Second, this test is generally more powerful than the standard projection tests as in Dufour (1997), Dufour and Jasiak (2001), Dufour et al. (2006), Dufour and Taamouti (2005, 2007), etc. Indeed, under the classical conditions as in Newey and McFadden (1994)'s Theorem 9.2 (henceforth NM-9.2) and a global identification condition for the nuisance subvector, this test is locally efficient. This efficiency, which is also the focus of our paper, is what fundamentally distinguishes the two-step test from the standard projection tests noted above, or the Bonferroni tests in Dufour (1990), Berger and Boos (1994), Silvapulle (1996), etc.

In this paper, we establish that all these results of Chaudhuri and Zivot (2011) remain valid for testing the hypothesis on general linear restrictions in (2) under more general characterization of the identification failure of θ^0. Importantly, the efficiency result is also strengthened to cover non-classical scenarios that do not satisfy the NM-9.2 conditions, and this is the main contribution of our paper.

As noted in the abstract and revisited in the sequel (Sections 2 (F2) and 4.2), the discussion of efficiency under non-classical scenarios involving rate-entangled identification of $R\theta^0$ leads to novel issues that, to our knowledge, have not been addressed elsewhere in this literature. We point out and address them under a unified framework, and also demonstrate them analytically and visually with the help of a simple example and a small-scale simulation study. More broadly, our discussion is related to the fundamentally important question: What is the benchmark for feasible local efficiency? The answer to this is obvious in Chaudhuri and Zivot (2011) and the related papers (see Section 2), but turns out to be more nuanced and *not obvious* under our setup since we allow for rate-entangled identification of $R\theta^0$.

The rest of the paper is organized as follows. Section 2 summarizes at the outset the key features of our paper, their novelty, consequences, and relation to the literature. Section 3 describes the two-step test and heuristically discusses the efficiency result under the classical NM-9.2 setup. The key to efficiency is the test statistic used in the second step, and for this we use the (GMM) LM version of Neyman (1959)'s C-alpha statistic. We show its numerical equivalence with the efficient score statistic used in Chaudhuri and Zivot (2011). While an equivalence result is not surprising thanks to the relation between the efficient score and influence functions, this numerical equivalence is new to our knowledge.

Section 4 allows for identification failure of θ^0 and treats the NM-9.2 setup as a special case. In Section 4.1, the two-step test's asymptotic rejection probability of the true null is discussed under the setup of Andrews and Guggenberger (2014). In Section 4.2, efficiency for suitable local deviations of the null from the truth is discussed by ruling out weak or worse identification of θ^0 but still allowing for multiple rates (strength) of identification in between weak and strong. For this, we impose more structure that nevertheless covers the setups of Stock and Wright (2000), Antoine and Renault (2012), etc. This efficiency consideration under rate-entangled identification of $R\theta^0$ is the focus of our paper.

Certain rather long but important definitions for Section 4.2 are collected in Appendix A. Technical proofs and expository materials are collected as Supplementary Materials in Appendices B, C and D.

2 Key features of our paper and the relation to the literature

To put the contributions of this paper into perspective, let us first briefly summarize the key features (F1)-(F4) of our paper, their consequences, and their relation to the literature.

(F1) Re-parameterize (1) **to facilitate imposing** H_0 **in the two-step test:** Guided by (2), consider a $(d_\theta - d_R)\times d_\theta$ matrix S such that the $d_\theta \times d_\theta$ matrix $A_S = [R', S']'$, indexed by S, is nonsingular. S exists since R is full row-rank. Now, for this S, consider the invertible linear transformation of θ:

$$(\beta', \gamma_S')' := A_S\theta. \tag{3}$$

(1) and (3) imply that $\beta^0 := R\theta^0$ and $\gamma_S^0 := S\theta^0$ are the true values for β and γ_S. The parameter space for $(\beta', \gamma_S')'$ is $\mathcal{B}\times\Gamma_S$ where $\mathcal{B} := \{R\theta : \theta\in\Theta\}\subset\mathbb{R}^{d_R}$ and $\Gamma_S := \{S\theta : \theta\in\Theta\}\subset\mathbb{R}^{d_\theta - d_R}$. The two-step test: (i) constructs a possibly restricted-by-H_0 confidence set for γ_S^0 in the first step, and (ii) rejects H_0 if either this confidence set is empty or if the second-step statistic, which is the LM C-alpha statistic

fixed at $\beta = r_0$ and minimized over γ_S in the confidence set, exceeds a pre-chosen critical value. Given our numerical equivalence result in Section 3, this is exactly same as the two-step projection test for $H_0 : \beta = r_0$ in Chaudhuri and Zivot (2011) designed specifically under the re-parameterized system (3).

(F2) Describe the framework in terms of the original parameter θ: Note that, in practice, moment restrictions such as (1) are typically conceived in terms of θ, and not $\beta := R\theta$ or an ad-hoc nuisance parameter $\gamma_S := S\theta$. Hence, we deviate from Chaudhuri and Zivot (2011) in that we use (3) only for the computational ease of imposing H_0, while we describe the framework entirely in terms of θ, more precisely, θ^0, by strictly adhering to (1).[1] Since we allow for multiple rates of identification for the elements of θ^0, the novel issues in the efficiency consideration arise precisely because such allowance can make, *concurrently*, the identification of β^0 rate-entangled and that of γ_S^0 dependent on the choice of S.[2]

To elaborate, let $d_\theta = 2$, $\theta = (\theta_1, \theta_2)'$ and $R = [1, 1]$, i.e., $\beta = \theta_1 + \theta_2$. Consider a γ_S (e.g., $\gamma_S = \theta_1$ if $S = [1, 0]$, $\gamma_S = \theta_2$ if $S = [0, 1]$) for (3). The local efficiency of the two-step test appeals to its asymptotic equivalence with an infeasible LM test that skips the first step and instead plugs-in the unknown true γ_S^0 (hence, infeasible), *regardless of* H_0, in the LM C-alpha statistic for the second step. Crucially, the infeasible plug-in test is not invariant to S unless H_0 is true. Indeed, it can be very different even locally for the different choices of S if the strengths of identification for the corresponding γ_S^0's are not the same. On the other hand, the two-step test is generally invariant to S. Hence, the interpretation of the local deviations of H_0 from the truth, for which the said asymptotic equivalences hold, needs particular care.

For example, let θ_1^0 be nearly strongly and θ_2^0 strongly identified. Then, fixing $\gamma_S = \theta_1$ at θ_1^0 by an infeasible test leads the local deviation in $\beta = \theta_1 + \theta_2$ to be along the strong direction θ_2, while fixing $\gamma_S = \theta_2$ at θ_2^0 by an infeasible test leads the deviation to be along the nearly strong directions θ_1.

In this context, our results in Section 4.2 reflect a novel phenomenon: The two-step test's asymptotic equivalence with the former infeasible test ($\gamma_S = \theta_1$), which is the more powerful one, holds more locally in terms of the local deviation than that with the latter infeasible test ($\gamma_S = \theta_2$). This is intuitive since the more powerful infeasible test assumes knowledge of the less strongly identified nuisance parameter ($\gamma_S = \theta_1$) and, therefore, it is impossible for the two-step test, which is feasible since it cannot make

[1]Given this importance that we assign to θ^0, we let the representation in (1) and (2) suffer from two drawbacks that hinder a satisfactory treatment of issues related to similarity and, hence, size of the two-step test. First, (1) and (2) do not adequately distinguish between the truth for θ (i.e., $\theta = \theta^0$), and what would be the truth for β (i.e., $\beta = \beta^0$, equivalently, $R\theta = R\theta^0$) in a standard subvector test representation. The former is a point in Θ, while the latter is a $(d_\theta - d_R)$ dimensional linear subspace of Θ. Second, we consider the true θ (i.e, θ^0) and hence, given the first drawback, the true β (i.e., β^0) as fixed but let the hypothesized value r_0 vary (possibly with T), which is what determines if H_0 is true or false. Accordingly, our assumptions focus on the fixed true θ^0. While both drawbacks could be bypassed by maintaining the setup in terms of β and γ_S (and for only the first one, by maintaining a global version of joint weak convergence assumptions of Kleibergen (2005)), given our focus on highlighting the issues related to local efficiency, we do not do so for brevity and instead refer the reader to Andrews (2017) for a comprehensive treatment along this route for the asymptotic size of two-step tests.

[2]While related, these issues are not the same as in Antoine and Renault (2009). Also, they do not arise in the study of efficiency with a single rate of identification as in Chaudhuri and Zivot (2011), (I.)Andrews (2016b), Andrews (2017), etc. They are also standard if assumptions on identification are instead maintained directly on β and the specific γ_S being used.

such assumptions, to resemble the behavior of such an infeasible test in a meaningfully large region.

(F3) Impose H_0 in the first-step confidence set: One might infer from (F2) and the non-invariance of the infeasible test that the two-step test could have better power for certain choices of S if H_0 is not imposed in step one. However, this has no justification since the two-step test may still be invariant to S. In fact, this leads to very poor power in small samples except in the NM-9.2 setup, in which case it is actually immaterial. So, this should not be done. On the other hand, this appears to better suit the LM plug-in test which then loses its invariance to S, and resembles the corresponding S-dependent infeasible test [c.f. Conniffe (2001), but in a different context]. Nevertheless, as shown in Appendix D, this leads to over-sized LM plug-in tests in cases where a standard, i.e., restricted-by-H_0, LM plug-in would have worked (e.g., cases covered by Theorem 6 of Guggenberger and Smith (2005)).

(F4) Our efficiency consideration is strictly local: The notion of optimality/efficiency in relation to the infeasible test is less ambitious than that considered in the literature on identification failure; e.g., Moreira (2003), Andrews et al. (2006), Chernozhukov et al. (2009), Moreira and Moreira (2013), (I.)Andrews (2016a), (I.)Andrews and Mikusheva (2016), Montiel-Olea (2016), etc. By contrast, our use of the term is similar to that in Section 9 of Andrews and Guggenberger (2015), or Comment (iii) following Theorem 4.1 of Andrews and Guggenberger (2014), or the oracle equivalence in Andrews (2017). Indeed, the LM-principle generally does not lead to optimality other than in a local sense since it is only based on the slope of the moment vector. Also, as originally noted by Kleibergen (2005), allowing for identification failure necessitates the use of an estimator for the Jacobian matrix that is not simply the sample mean of the derivative of the moment vector, but the sample mean of the residual of the regression of this derivative on the moment vector itself. In certain cases of identification failure, this affects the intended direction along which the LM-principle maximizes local power; see, e.g., Antoine and Renault (2009), (I.)Andrews (2016b), etc. Even otherwise, this may lead to a spurious decline in power away from the truth [see Kleibergen (2005)], which, however, is partially addressed by the two-step test by virtue of a specific choice of the first-step confidence set [see Chaudhuri and Zivot (2011)].

Related literature: While we generalize the use of the LM and C-alpha principle in Chaudhuri (2008), Zivot and Chaudhuri (2009), Chaudhuri et al. (2010) and Chaudhuri and Zivot (2011); the LM and/or C-alpha tests were originally used in the context of identification failure by Wang and Zivot (1998), Dufour and Jasiak (2001), Kleibergen (2002), Moreira (2003), Kleibergen (2005), Guggenberger and Smith (2005), Antoine and Renault (2009), etc. It has also been considered recently in Magnusson and Mavroeidis (2010), Guggenberger et al. (2012b), Qu (2014), Dufour et al. (2015, 2016), Andrews and Mikusheva (2015), Andrews and Guggenberger (2014), etc. Even more recently, McCloskey (2015) and Andrews (2016b) propose sophisticated related methods to improve the performance of such tests.

There are two papers that are closest to ours. Of them, we have already noted above the relation with Chaudhuri and Zivot (2011). The other paper is Andrews (2017), where the basic two-step test is substantially generalized, extended, refined and demonstrably improved. However, there, the efficiency result is not considered under the type of rate-entangled identification that leads to the key feature (F2). More precisely, although testing for even nonlinear restrictions is considered in Andrews (2017), efficiency is actually discussed for tests of genuine subvectors under: (i) strong local identification, i.e., a single rate, and (ii) variational independence of the subvectors being tested and not tested [c.f. (F2)].[3]

In contrast to Andrews (2017) and all the papers noted above (and with our focus strictly on the discussions of efficiency therein), we work with rate-entangled identification by allowing for multiple rates and also the variational dependence of the linear combinations being tested and not tested (as in the example in (F2)). Accordingly, we adopt a just-sufficiently general framework in Section 4.2 enabling us to highlight the ensuing local-efficiency-related key feature (F2) that is the novelty of our paper.

3 Definition and an overview of the improved two-step projection test

This section maintains the classical NM-9.2 setup as default. Define $\bar{g}_T(\theta) := \frac{1}{T}\sum_{t=1}^{T} g(Z_t;\theta)$, $G(\theta) := \frac{\partial}{\partial\theta'}E\left[g(Z_t;\theta)\right]$ and $V(\theta) := Var\left(g(Z_t;\theta)\right)$. Then, the efficient GMM estimator of $R\theta^0$ has the asymptotically linear representation: $\sqrt{T}(\widehat{R\theta^0} - R\theta^0) = -\sqrt{T}l_T(\theta^0) + o_p(1)$ [see Appendix B.1 for details] where

$$l_T(\theta) := R\left(G'(\theta)V^{-1}(\theta)G(\theta)\right)^{-1}G'(\theta)V^{-1}(\theta)\bar{g}_T(\theta),$$

if it exists. Therefore, for local optimality/efficiency, a test for H_0 in (2) can be based on a consistent estimator of $l_T(\theta)$:

$$\widehat{l}_T(\theta) := R\left(\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)\right)^{-1}\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\bar{g}_T(\theta)$$

where $\widehat{G}_T(\theta) \xrightarrow{P} G(\theta)$ and $\widehat{V}_T(\theta) \xrightarrow{P} V(\theta)$. While the inverse in the above expression may not exist for a given T, we will maintain conditions such that its probability limit exists either without (in Section 3) or with (in Section 4) an appropriate scaling. This is facilitated by our asymmetric treatment of $G(\theta)$ and $V(\theta)$, in the sense that we will never allow for any rank-failure for $V(\theta)$ [see Andrews and Guggenberger (2015) for such cases] but will allow for rank-failure of $G(\theta)$ in Section 4.

Guided by the consideration of local efficiency and the above representation of $\widehat{l}_T(\theta)$, we use the following quadratic form of $\widehat{l}_T(\theta)$ for the second step of the two-step test:

[3]If the linear restrictions $R\theta^0 = r_0$ are replaced by nonlinear restrictions, say, $R(\theta^0) = 0$ then there typically exist functions $S(\theta)$ such that $A_S(\theta^0) = (R'(\theta^0), S(\theta^0)')'$ is invertible. However, for simplicity of exposition without obfuscating our main message by issues such as a θ-dependent S, we choose to work with linear restrictions in this paper. As a result, we only have to employ a θ-dependent S once — in an intermediate step of the proof of our Lemma 2, but nowhere else. In cases of nonlinear $R(\theta)$, such an $S(\theta)$ for this intermediate step typically exists in a tubular neighborhood of the given θ.

$$
\begin{aligned}
LM_T(\theta) &:= T \times \widehat{l}_T'(\theta)\left(R\left(\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)\right)^{-1}R'\right)^{-1}\widehat{l}_T(\theta) \\
&= T \times \left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)' P\left(\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)\left(\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)\right)^{-1}R'\right)\left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)
\end{aligned} \tag{4}
$$

where we use the notation $P(D) := D(D'D)^{-1}D'$ to define the projection matrix for any matrix D with the same caveat about the existence of the inverse as noted above (4); and if D is positive semidefinite then we define $D^{1/2}$ to be the upper triangular matrix such that $D = D^{1/2'}D^{1/2}$.

$LM_T(\theta)$ is Smith (1987)'s LLM_T, Dagenais and Dufour (1991)'s PC or Newey and McFadden (1994)'s LM_{2n} statistic for testing linear restrictions. It falls under the class of Neyman (1959)'s C-alpha statistic.

Note that, (3) and (4) give:

$$
LM_T(\theta) \equiv LM_T\left(A_S^{-1}(\beta', \gamma_S')'\right). \tag{5}
$$

Definition: For $\epsilon, \alpha > 0$ such that $\epsilon + \alpha < 1$, the improved two-step GMM-LM projection test, or simply the two-step test, for H_0 in (2) is defined as:

$$
\begin{aligned}
&\text{Step 1: obtain a nominal } (1-\epsilon)\text{-level confidence set } CI_T(\gamma_S;\epsilon) \text{ for } \gamma_S^0; \\
&\text{Step 2: reject } H_0 \text{ if } CI_T(\gamma_S;\epsilon) \text{ is empty or if } \inf_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0', \gamma_0')'\right) > \chi^2_{d_R}(1-\alpha)
\end{aligned} \tag{6}
$$

where $\chi^2_{d_R}(1-\alpha)$ is the $(1-\alpha)$-th quantile of a central χ^2 distribution with d_R degrees of freedom.

$CI_T(\gamma_S;\epsilon)$ is what we refer to as the first-step confidence set, i.e., the preliminary non-point (set) estimator of the nuisance parameter $\gamma_S^0 := S\theta^0$ that is not specified by H_0. As noted below, it plays an important role in influencing the asymptotic properties and the computational ease of the two-step test.

Remark 1: Invariance of the two-step test to the choice of S in (3) is preserved by the conventional confidence sets $CI_T(\gamma_S;\epsilon)$ regardless of the non-uniqueness of the infimum in the second step. If possible, however, choosing an S with a better identified γ_S^0 might help with the computation in the second step.

Remark 2: To accommodate for identification failures of θ^0, $CI_T(\gamma_S;\epsilon)$ can be obtained by inverting, e.g., the S-test, the K-test, modifications of Moreira (2003)'s CLR test [see Kleibergen (2005), Andrews and Guggenberger (2014, 2015)] for γ_S, while treating $\beta = r_0$ as known. In practice, the operations required in steps one and two can be simultaneously conducted since, to fail to reject H_0, it is sufficient to find a single point γ_0 that would belong in $CI_T(\gamma_S;\epsilon)$ and also satisfy the condition for step two.

Remark 3: It is sufficient to focus on the second-step test statistic to see the connection between (6) and the two-step projection subvector test in Chaudhuri and Zivot (2011), i.e., to see that the test in (6) is the natural extension of Chaudhuri and Zivot (2011). To this end, for a given S, let R_S^1 and S_S^1 be the $d_\theta \times d_R$ and $d_\theta \times (d_\theta - d_R)$ matrices respectively such that $A_S^{-1} = [R_S^1, S_S^1]$. Then, in this context, the test statistic in Chaudhuri and Zivot (2011)'s second step would be [see Appendix B.2 for details]:

$$LM_{T,S}^{\mathrm{ES}}(\theta) := T \times \left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)' P\left(\left(I_{d_g} - P\left(\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)S_S^1\right)\right)\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)R_S^1\right)\left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right),$$

which they refer to as the efficient score (ES) statistic for $\beta = R\theta$. We prove in Appendix B.3 that:

Lemma 1 *Let $\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)$ be positive definite for a given T and $\theta \in \Theta$. Then $LM_{T,S^*}^{ES}(\theta) = LM_{T,S^\dagger}^{ES}(\theta)$ for any $S = S^*, S^\dagger$ for which $A_S := [R', S']'$ is nonsingular.*

Lemma 2 *Let $\widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)$ be positive definite for a given T and $\theta \in \Theta$. Then $LM_T(\theta) = LM_{T,S}^{ES}(\theta)$ for any choice of S for which $A_S := [R', S']'$ is nonsingular.*

Lemma 1 builds on Dagenais and Dufour (1991) and establishes invariance for $LM_{T,S}^{\mathrm{ES}}(\theta)$ similar to that for $LM_T(\theta)$ in (5). Lemma 2 reconciles between $LM_T(\theta)$ and $LM_{T,S}^{\mathrm{ES}}(\theta)$ and shows that they are the same in the context of testing (2) with a general R. While expected from the relationship between the efficient influence function and the efficient score function, on which $LM_T(\theta)$ and $LM_{T,S}^{\mathrm{ES}}(\theta)$ are based respectively, this is, to our knowledge, a new result in the C-alpha literature [see, e.g., Smith (1987), Dagenais and Dufour (1991), Bera and Bilias (2001), Dufour et al. (2015, 2016), Andrews (2017), and also the references therein for the related papers in the statistics literature].

Remark 4: It is useful to note here that the conventional projection test rejects H_0 at the level α if:

$$\inf_{\theta_0 \in \Theta : R\theta_0 = r_0} \widetilde{LM}_T(\theta_0) > \chi^2_{d_\theta}(1-\alpha) \quad \text{or equivalently,} \quad \inf_{\gamma_0 \in \Gamma_S} \widetilde{LM}_T\left(A_S^{-1}(r_0', \gamma_0')'\right) > \chi^2_{d_\theta}(1-\alpha) \qquad (7)$$

where [see the LM_{3n} statistic in Newey and McFadden (1994) and the K statistic in Kleibergen (2005)]:

$$\widetilde{LM}_T(\theta) := T \times \left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)' P\left(\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)\right)\left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right).$$

The second version in (7) explicitly imposes H_0 by using (3) and thus streamlines the computation for the test. Note that, $\widetilde{LM}_T(\widetilde{\theta}_T) = LM_T(\widetilde{\theta}_T)$ where $\widetilde{\theta}_T$ is the restricted-by-H_0 GMM estimator of θ_T [see Appendix B.4]. That is, $\widetilde{LM}_T(\widetilde{\theta}_T) = LM_T(\widetilde{\theta}_T)$ is the standard plug-in LM statistic for testing H_0 in (2). Then, NM-9.2 gives: $LM_T(\widetilde{\theta}_T) \xrightarrow{d} \chi^2_{d_R}$ distribution, which is central if H_0 is true, and non-central under local deviations of H_0. Hence, the conservativeness (and thereby, the inefficiency) of the conventional projection test, that the two-step test would address, is due to the use of a $\chi^2_{d_\theta}$ critical value in (7), while the test statistic is actually $\inf_{\theta_0 \in \Theta : R\theta_0 = r_0} \widetilde{LM}_T(\theta_0) \le \widetilde{LM}_T(\widetilde{\theta}_T) = LM_T(\widetilde{\theta}_T) \xrightarrow{d} \chi^2_{d_R}$.

Remark 5: By contrast, the local efficiency of the two-step test, that one would expect from (4), results as follows. For the given S, let $\theta_0 := A_S^{-1}(r_0', \gamma_0')'$ where $r_0 := \beta^0 + \mu_\beta/\sqrt{T}$, $\gamma_0 := \gamma_S^0 + \mu_{\gamma_S}/\sqrt{T}$, μ_β is a constant and $\mu_{\gamma_S} = O_p(1)$. Then, NM-9.2 gives: $\widehat{G}_T(\theta_0) \xrightarrow{P} G(\theta^0)$, $\widehat{V}_T(\theta_0) \xrightarrow{P} V(\theta^0)$ and, crucially,

$$\sqrt{T}\widehat{l}_T(\theta_0) \xrightarrow{P} \sqrt{T}l_T(\theta^0) + \mu_\beta.$$

So, NM-9.2 gives: $LM_T(\theta_0) \xrightarrow{d} \chi^2_{d_R}$ with non-centrality parameter $\mu_\beta' \left(R\left(G'(\theta^0)V^{-1}(\theta^0)G(\theta^0)\right)^{-1} R'\right)^{-1} \mu_\beta$, which does not depend on the $\sqrt{T}$-deviation of γ_0 from γ_S^0. On the other hand, under the same conditions and a global strong identification condition for γ_S^0 given β^0, it can be shown following Andrews (2017) that:

$$\sup_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} \sqrt{T}\|\gamma_0 - \gamma_S^0\| = O_p(1)$$

for conventional confidence sets, provided that they are non-empty (to fix ideas for now). Hence, by construction, $\gamma_{S,T}^\dagger := \arg\inf_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0', \gamma_0')'\right) = \gamma_S^0 + \mu_{\gamma_S,T}/\sqrt{T}$ for some $\mu_{\gamma_S,T} = O_p(1)$. Therefore, the two-step test in (6), with a non-empty first-step confidence set, is locally efficient in the sense of (F4) [see Section 2] since it is asymptotically equivalent to the locally optimal/efficient infeasible test — infeasible, since it is based on the unknown true γ_S^0 — that rejects H_0 at the level α if for $\theta_{0,S}^{infs} := A_S^{-1}(r_0', \gamma_S^{0\prime})'$:

$$LM_T(\theta_{0,S}^{infs}) > \chi^2_{d_R}(1-\alpha). \tag{8}$$

This optimality discussion is only under the classical NM-9.2 setup. Section 4 presents a general treatment allowing for identification failure of θ^0 and, hence, considering the NM-9.2 setup as a special case.

4 Asymptotic Properties: When identification failure of θ^0 is allowed

Now, we allow for possible identification failure of θ^0 due to possible rank-failure of $G(\theta^0)$. Then, it is imperative that the choice of $\widehat{G}_T(\theta)$ in the definition of $LM_T(\theta)$ in (4) follows Kleibergen (2005). That is:

$$\widehat{G}_T(\theta) := \left[\widehat{G}_{1,T}(\theta), \ldots, \widehat{G}_{d_\theta,T}(\theta)\right] \quad \text{where} \quad \widehat{G}_{T,j}(\theta) := \frac{\partial}{\partial\theta_j}\bar{g}_T(\theta) - \widehat{V}_{j,g,T}(\theta)\widehat{V}_T^{-1}(\theta)\bar{g}_T(\theta),$$

$\widehat{V}_{j,g,T}(\theta)$ and $\widehat{V}_T(\theta)$ are respectively $d_\theta \times d_g$ and $d_g \times d_g$ matrices, and θ_j is the j-th element of θ for $j = 1, \ldots, d_\theta$. Unless otherwise noted, e.g., Remark 9, we require $\widehat{V}_{j,g,T}(\theta)$ and $\widehat{V}_T(\theta)$ to be estimators of

$$\begin{aligned} V_{j,g}(\theta) &:= \lim_{T\to\infty} T \times E\left[\left(\frac{\partial}{\partial\theta_j}\bar{g}_T(\theta) - E\left[\frac{\partial}{\partial\theta_j}\bar{g}_T(\theta)\right]\right)\bar{g}_T(\theta)'\right] \text{ for } j = 1, \ldots, d_\theta \\ \text{and } V(\theta) &:= \lim_{T\to\infty} T \times E\left[\left(\bar{g}_T(\theta) - E[\bar{g}_T(\theta)]\right)\bar{g}_T(\theta)'\right], \end{aligned}$$

provided that they exist. Also applicable are the choices of $\widehat{G}_T(\theta)$ considered in Guggenberger and Smith (2005, 2008) that only deviate from $\widehat{G}_T(\theta)$ defined above by an order of magnitude of $o_p(1/\sqrt{T})$.

We maintain high-level but standard assumptions on the joint distribution F_T of the data $\{Z_t\}_{t=1}^T$. Allowing for a drifting data generating process (DGP) in what follows is important, and to emphasize it we index by T the key parameters defined in terms of F_T; see, e.g., Stock and Wright (2000), Andrews and Guggenberger (2014). However, irrespective of the drifting DGP $\{F_T : T \geq 1\}$, we take the truth θ^0 satisfying the moment restrictions in (1) as fixed. H_0 in (2) is true if the hypothesized value r_0 is equal to $R\theta^0$, it is false otherwise. Apart from characterizing the false H_0 by locally deviating (made precise later in (14) and (17)) r_0 from $R\theta^0$, no other assumptions involve r_0. For convenience, we maintain that:

Assumption O:

$\theta^0 \in \text{interior}(\Theta)$ where Θ is compact in $\mathbb{R}^{d_\theta}$.

Notation: We suppress the triangular array $\{Z_{t,T} : t = 1, \ldots, T; T \geq 1\}$ notation, and instead denote $Z_{t,T}$ by Z_t. $\underline{c} > 0$ and $\bar{c} > 0$ will denote generic constants. For any matrix D, define $\|D\| := \sqrt{\text{trace}(D'D)}$. For any $a \times b$ matrix $D = [D_1, \ldots, D_b]$ define $D_{(j:k)} := [D_j, \ldots, D_k]$ as the $a \times (k - j + 1)$ matrix for $1 \leq j \leq k \leq b$. $D_{(k:j)}$ is an empty matrix for $0 \leq j < k \leq b+1$. For an $(ab) \times 1$ vector $D = (d_1, \ldots, d_{ab})'$, define $devec_b(D) := [(d_1, \ldots, d_b)', (d_{b+1}, \ldots, d_{2b})', \ldots, (d_{(a-1)b+1}, \ldots, d_{ab})']$ as a $b \times a$ matrix.

4.1 Rejection of the null hypothesis in (2) when it is true

Assumption M:

M1. $\frac{\partial}{\partial \theta'} g(z; \theta^0)$ exists for each $z \in \mathbb{R}^{d_z}$. Let $\bar{G}_T(\theta) := \frac{1}{T} \sum_{t=1}^T \frac{\partial}{\partial \theta'} g(Z_t; \theta)$ and $G_T := E_T[\bar{G}_T(\theta^0)]$. Then,

$$\left[\sqrt{T}\bar{g}_T'(\theta^0),\ \sqrt{T}vec(\bar{G}_T(\theta^0) - G_T)'\right] \xrightarrow{d} [\psi',\ \psi_G'] \sim N(0, \Sigma)$$

$$\text{and } \lim_{T\to\infty} Var_T \begin{pmatrix} \sqrt{T}\bar{g}_T(\theta^0) \\ \sqrt{T}vec(\bar{G}_T(\theta^0)) \end{pmatrix} \equiv \lim_{T\to\infty} \begin{bmatrix} V_T & V_{gG,T} \\ V_{Gg,T} & V_{GG,T} \end{bmatrix} = \Sigma := \begin{bmatrix} V & V_{gG} \\ V_{Gg} & V_{GG} \end{bmatrix}.$$

M2. $\|G_T\|, \|V_T\|, \|V_T^{-1}\|, \|V_{Gg,T}\| \leq \bar{c}$ for all T. $\|\widehat{V}_T(\theta^0) - V_T\| = o_p(1)$ and $\|\widehat{V}_{Gg,T}(\theta^0) - V_{Gg,T}\| = o_p(1)$.

To characterize any identification failure of θ^0, consider the singular value decomposition of $V_T^{-1/2}G_T$:

$$V_T^{-1/2}G_T = C_T\bar{\Delta}_T B_T' \tag{9}$$

where C_T and B_T are $d_g \times d_g$ and $d_\theta \times d_\theta$ orthogonal matrices whose columns are respectively the eigenvectors of the matrices $V_T^{-1/2}G_T G_T' V^{-1/2'}$ and $G_T' V^{-1} G_T$. $\bar{\Delta}_T := [\Delta_T, 0]'$ is the $d_g \times d_\theta$ matrix where $\Delta_T := \text{diag}\left(\delta_{T,1}, \ldots, \delta_{T,d_\theta}\right)$ is the $d_\theta \times d_\theta$ diagonal matrix with its diagonal elements $\delta_{T,1} \geq \delta_{T,2} \geq \ldots \geq \delta_{T,d_\theta}$ $(\geq 0,$ without loss of generality$)$ as the singular values of $V_T^{-1/2}G_T$.

Assumption M (continued)**:** (identification failure of θ^0 following Andrews and Guggenberger (2014))

M3. For the singular value decomposition in (9), there exists a $p \in \{0, 1, \dots, d_\theta\}$ such that:

(a) $\delta_{T,j} \to \delta_j$, a constant, and $\sqrt{T}\delta_{T,j} \to \infty$ for $j = 1, \dots, p$ as $T \to \infty$ (M3(a) is void if $p = 0$).

(b) $\sqrt{T}\delta_{T,j} \to l_j$, a constant, for $j = p+1, \dots, d_R$ as $T \to \infty$ (M3(b) is void if $p = d_\theta$).

(c) $C_T \to C$ and $B_T \to B$ as $T \to \infty$ where B is a nonsingular matrix.

(d) If $p < d_\theta$, then $G^* := [C_{(1:p)}, C_{(p+1:d_\theta)}L + V^{-1/2}(\theta^0)devec_{d_g}(\psi_G - V_{Gg}V^{-1}\psi)B_{(p+1:d_\theta)}]$ is a $d_g \times d_\theta$ matrix with full column-rank d_θ almost surely, where $L := \text{diag}\,(l_{p+1}, \dots, l_{d_\theta})$ is a $(d_\theta - p) \times (d_\theta - p)$ diagonal matrix with $l_{p+1}, \dots, l_{d_\theta}$ as its diagonal elements. Else $G^* := C_{(1:p)}$.

Remark 6: p is the number of directions in θ that are better than weakly identified. The remaining $d_\theta - p$ directions in θ are at best weakly identified, which necessitates the particular choice of $\widehat{G}_T(\theta)$. Assumption M3 and the representation involved in it are entirely based on the original work of Andrews and Guggenberger (2014). M3 is actually slightly stronger than what Andrews and Guggenberger (2014) require. This helps to avoid certain peripheral complications arising from the fact that $d_R < d_\theta$. Assumptions O, M1 and M2 are standard; see, e.g., Kleibergen (2005), Guggenberger and Smith (2005).

Lemma 3 *Let assumptions O and M1-M3 hold. Then, for $LM_T(\theta^0)$ defined in (4), $LM_T(\theta^0) \xrightarrow{d} \chi^2_{d_R}$.*

Proposition 4 *Let the null hypothesis H_0 in (2) be true, i.e., $r_0 = R\theta^0$ for θ^0 defined in (1). Let the joint distribution $\{F_T : T \geq 1\}$ of $\{Z_t\}_{t=1}^T$ be constrained by the assumptions O and M1-M3. Let $\epsilon, \alpha > 0$ and $\epsilon + \alpha < 1$. Let $CI_T(\gamma_S; \epsilon)$ be a confidence set for γ_S defined in (3) with asymptotic coverage $(1 - \epsilon)$ for $\gamma_S^0 := S\theta^0$. Then, the probability with which the improved two-step projection test in (6) rejects H_0 cannot exceed $(\epsilon + \alpha)$ asymptotically.*

Remark 7: The result follows by Bonferroni's inequality applied to Lemma 3 and the asymptotic coverage of $CI_T(\gamma_S; \epsilon)$. Importantly, the upper bound $(\epsilon + \alpha)$ is entirely under the control of the user.

Remark 8: An example of the first-step confidence set $CI_T(\gamma_S; \epsilon)$ that possesses this coverage property is:

$$CI_T^{SW}(\gamma_S; r_0, \epsilon) := \left\{\gamma_0 \in \Gamma_S : T \times Q_T(A_S^{-1}(r_0', \gamma_0)') \leq \chi^2_{d_g}(1 - \epsilon)\right\}. \tag{10}$$

It is obtained by inverting the S-test of Stock and Wright (2000) (SW). r_0 is the hypothesized value from H_0 in (2). $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ imposes H_0 and utilizes the re-parameterization in (3) for the computation of:

$$Q_T(\theta) := \bar{g}_T'(\theta)\widehat{V}_T^{-1}(\theta)\bar{g}_T(\theta), \tag{11}$$

the continuously updated (CU) GMM criterion function. By Theorem 2 of Stock and Wright (2000), the asymptotic coverage of $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ for $\gamma_S^0 := S\theta^0$ is $(1 - \epsilon)$ when H_0 in (2) is true and also: (a)

$\sqrt{T}\bar{g}_T(\theta^0) \xrightarrow{d} \psi$ and (b) $\widehat{V}_T(\theta^0) \xrightarrow{P} V$. Since (a) and (b) are included in M1 and M2, the asymptotic coverage for $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ remains $(1-\epsilon)$ under weaker conditions than what we maintain here.

What also works when H_0 is true, but not recommended otherwise since it generally leads to very poor power except under NM-9.2 setup [see (F3) in Section 2, and also Appendix D], is:

$$\text{an unrestricted-by-}H_0\text{ version: } CI_T^{ur-SW}(\gamma_S; \epsilon) := \left\{\gamma = S\theta : \theta \in \Theta,\ T \times Q_T(\theta) \leq \chi^2_{d_g}(1-\epsilon)\right\}. \quad (12)$$

We have always found $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in (10) to be the most useful in practice for all purposes.[4,5]

4.2 Rejection of the null hypothesis in (2) when it locally deviates from the truth

We focus on constructing local deviations of the null from the truth that make the null locally false, and generate results resembling that in the classical NM-9.2 setup. While the construction is kept generic, as noted in Section 2 (F2), its interpretation for specific choices of S in (3) helps to clarify the local efficiency properties of the two-step test. This is a key aspect of this subsection. To appeal to contiguity arguments, we rule out weak or worse identification of θ^0. In terms of assumption M3, it means $p = d_\theta$.

Given this, a key issue is the rate at which $CI_T(\gamma_S; \epsilon)$ shrinks to γ_S^0 [see Remark 5]. This necessitates characterizing $E_T[\bar{g}_T(\theta)]$ globally for $\theta \in \Theta$ (more specifically, $E_T[\bar{g}_T(A_S^{-1}(r_0', \gamma_S')')]$ for $\gamma_S \in \Gamma_S$).[6] Two well-known setups for this are Stock and Wright (2000) and Antoine and Renault (2012). They differ in terms of their consequences in that the former results in rate-disentangled local identification of θ^0, while the latter generally leads to rate-entanglement. Both will lead to rate-entangled $R\theta^0$ in general.

Both setups (and also that in Section 4.1) share a common trait that the local behavior of $E_T[\bar{g}_T(\theta)]$ under them can be characterized by the existence of a nonsingular matrix (depends on T and the setup) which, when *pre-multiplied by* the Jacobian $E_T[\bar{G}_T(\theta^0)]$, converges to a finite $d_g \times d_\theta$ matrix of full column-rank. Hence, the local analysis under both setups are similar except that Antoine and Renault (2012) demand a little extra work since it also makes the identification of θ^0 itself rate-entangled.

[4] Amongst the well-known identification-robust confidence sets [see Remark 2], Chaudhuri and Zivot (2011) recommend the use of $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ because of its: (i) validity under weak and general conditions, (ii) computational simplicity, and (iii) effectiveness in eliminating certain spurious declines in power of the GMM-LM test from the second step of the improved projection test. The ϵ in the upper bound in Proposition 4 is, in practice, primarily due to the fact that $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ can be empty with nonzero probability (that increases with ϵ). While possibly unsatisfying in theory [see Andrews (2017) for ways to avoid it], this feature is actually useful for (iii) and also for (ii), and hence is accommodated in the definition of the improved two-step projection test in (6). Thus, the recommendation is in spite of the concern raised in Davidson and MacKinnon (2014) and Muller and Norets (2016) (page 2184) that $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ does not properly reflect the parameter uncertainty. (This concern is at least partly addressed by the second step of the improved two-step projection test.)

[5] This is a continuation of footnote 4. While it is clear that $CI_T(\gamma_S; \epsilon)$ based on Kleibergen (2005)'s GMM-LM principle cannot be helpful for (iii) in general, it should be noted that $CI_T(\gamma_S; \epsilon)$ based on Moreira (2003)'s conditional likelihood ratio principle may not also be helpful for (iii). Simulation evidence and discussion on this can be found in Section 7.2.1 of Andrews (2016b). Both such $CI_T(\gamma_S; \epsilon)$'s can also be less appealing in terms of (i) and (ii) [see Mikusheva (2010) for (ii)].

[6] See Remark 5. Andrews (2017) ensures this in a similar setup by maintaining a global strong-identification condition SI2(i)/(13.2) for γ_S^0 given $\beta = r_0$, and a local strong-identification condition SI2(ii)/SS2$_{\text{LM}}$(i) for θ^0. By contrast, our local and global conditions are going to be inter-related, and they allow for worse than strong but better than weak identification.

Therefore, we take the following strategy: We follow Antoine and Renault (2012) for the discussion in this subsection, and remark on the simplifications that are possible under Stock and Wright (2000). Since modeling $E_T[\bar{g}_T(\theta)]$ has implications on its derivative and since local efficiency considerations rule out weak/joint-weak identification anyway, the general setup from Section 4.1 that focuses only on the Jacobian is now less indispensable. Hence, we abandon it. (See Andrews (2017) for a rigorous study of efficiency with that setup under some form of strong global and local identification [also see footnote 6].)

Our strategy leads to a framework that is (a little more than) just-sufficiently general for us to highlight the key feature (F2) related to the local efficiency of tests under rate-entangled identification and the variational dependence of the linear combinations of parameters being tested and not tested.

Accordingly, following Antoine and Renault (2012), for some $\rho : \Theta \mapsto \mathbb{R}^{d_g}$ and a sequence of diagonal matrices $\{\Lambda_T : T \geq 1\}$, let

$$E_T\left[\bar{g}_T(\theta)\right] = \frac{\Lambda_T}{\sqrt{T}}\rho(\theta). \tag{13}$$

Notation: Let 1_c denote the $1 \times c$ vector with all elements equal to 1. For a set of d_j-dimensional vectors $\{a_j\}_{j=1}^q$, let $\text{diag}(a_1, \ldots, a_q)$ denote the $\sum_{j=1}^q d_j$-dimensional diagonal matrix with diagonal elements as the elements of $a_1, \ldots, a_q$ respectively. Let $\mathcal{N}(\theta^0) \subset \Theta$ denote a generic open neighborhood of θ^0.

Assumption N: (following Antoine and Renault (2012))

N1. $\rho(\theta^0) = 0$ and $\inf_{\|\theta-\theta^0\|>\underline{c}} \|\rho(\theta)\| > 0$ for any $\underline{c} > 0$.

N2. $\psi_T(\theta) := \sqrt{T}\left(\bar{g}_T(\theta) - E_T\left[\bar{g}_T(\theta)\right]\right) \Rightarrow \psi(\theta)$ where $\psi(\theta)$ is a Gaussian process on Θ with mean zero and covariance function $V(\theta_1, \theta_2)$. $V(\theta^0) = V$ (as in M1) where $V(\theta) := V(\theta, \theta)$.

N3. $\{\Lambda_T : T \geq 1\}$ is a deterministic sequence of $d_g \times d_g$ diagonal matrix with positive diagonal elements. I^* is a $d_g \times d_g$ matrix whose rows are a suitable permutation of the rows of I_{d_g} such that $I^*\Lambda_T I^{*'} = \text{diag}(\lambda_{T,1}1_{k_1}, \ldots, \lambda_{T,l}1_{k_l})$ where $k_j > 0$ for $j = 1, \ldots, l$ and $\sum_{j=1}^l k_j = d_g$. $\lambda_{T,j} = o(\lambda_{T,j+1})$ for $j = 1, \ldots, l-1$. $\lim_T \lambda_{T,1} = \infty$, i.e., the worst rate is better than weak, but $\lim_T \lambda_{T,l}/\sqrt{T} < \infty$.[7]

N4. The $d_g \times d_\theta$ matrix $\rho_\theta(\theta) := \frac{\partial}{\partial\theta'}\rho(\theta)$ exists, has full column-rank d_θ, and is continuous in $\theta \in \mathcal{N}(\theta^0)$.

N5. $g(z; \theta)$ is differentiable in $\theta \in \mathcal{N}(\theta^0)$ for each $z \in \mathbb{R}^{d_z}$.

N6. $\frac{\partial}{\partial\theta'}\psi_T(\theta^0) = \sqrt{T}\left[\bar{G}_T(\theta^0) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\right] = O_p(1)$.

N7. (N7 (a) and (b) are grouped together to allow us to briefly discuss a tradeoff in Remark 9 below.)

   (a) $\rho(\theta)$ is twice continuously differentiable in $\theta \in \mathcal{N}(\theta^0)$. $g(z; \theta)$ is twice differentiable in $\theta \in \mathcal{N}(\theta^0)$ for each $z \in \mathbb{R}^{d_z}$. $\sup_{\theta\in\mathcal{N}(\theta^0)} \left\| \frac{\partial}{\partial\theta_i}\left[\bar{G}_T(\theta) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right] \right\| = o_p(\lambda_{T,l}/\sqrt{T})$ for $i = 1, \ldots, d_\theta$.

   (b) $\lambda_{T,1}^2/\lambda_{T,l} \to \infty$ as $T \to \infty$.

[7] $I^{*-1} = I^{*'}$. I^* is not unique unless $k_1 = \ldots = k_l = 1$ and thus $l = d_g$. The multiplicity of the elements can be made dependent on T and θ at the cost of significantly involved notation, but such generalizations may not be relevant in practice.

N8. $\sup_{\theta\in\Theta}\|\widehat{V}_T(\theta)-V(\theta)\| = o_p(1)$. $\sup_{\theta\in\mathcal{N}(\theta^0)}\|\widehat{V}_{Gg,T}(\theta)-V_{Gg}(\theta)\| = o_p(1)$. $V(\theta)$ and $V_{Gg}(\theta)$ are continuous in $\theta \in \mathcal{N}(\theta^0)$. $\sup_{\theta\in\Theta}\max[\text{eigen values}(V(\theta))] \leq \bar{c} < \infty$ and $\inf_{\theta\in\Theta}\min[\text{eigen values}(V(\theta))] \geq \underline{c} > 0$. Also, $V_{Gg}(\theta)$ is finite for $\theta \in \mathcal{N}(\theta^0)$.

Remark 9: Assumption N describes the setup following Antoine and Renault (2012) who also provide discussion on each of them. N8 deviates from M2 by abstracting from the $\{F_T\}$-(i.e., T)-dependence of the second moments $V_T(\theta)$ and $V_{Gg,T}(\theta)$ but does not otherwise interfere with the focus of our paper.

However, unlike in Antoine and Renault (2012), we would like to note the following points here. First, there is a tradeoff between the smoothness assumption N7(a) and the rate assumption in N7(b). This explains why, e.g., in a linear instrumental variables regression (explored further in Section 4.2.3), where the higher order derivatives of the moment vector are necessarily zero, there is no need to impose a lower bound to the rate at which $\lambda_{T,1} \to \infty$, i.e., any rate that is better than weak works irrespective of the best rate $\lambda_{T,l}$ [c.f. N7(b)].[8] Second, while $V_{Gg}(\theta)$ in N8 should ideally be such that $V_{Gg}(\theta^0) = V_{Gg}$ defined in M1, this is not necessary here since we no longer allow for weak identification in this subsection.

Remark 10: Under this setup, Antoine and Renault (2012) characterize an orthogonal rotation of θ^0 with rate-disentangled identification as $\Pi'_{\rho_\theta}\theta^0$ and with the rates given by $\sqrt{T}D^{-1}_{T,\rho_\theta}$. Π_{ρ_θ} and D_{T,ρ_θ} are, respectively, $d_\theta \times d_\theta$ orthogonal and diagonal matrices dependent on $\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)$ and with rather involved definitions that, for the sake of readability, are presented in detail in Appendices A.2.1 and A.2.2.

The intuition behind the matrices Π_{ρ_θ} and D_{T,ρ_θ} is that they allow us to write for any θ:

$$I^*E_T\left[\bar{G}_T(\theta^0)\right]\sqrt{T}(\theta-\theta^0) = \left\{I^*E_T\left[\bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta}D_{T,\rho_\theta}\right\}\left\{\sqrt{T}D^{-1}_{T,\rho_\theta}\Pi'_{\rho_\theta}(\theta-\theta^0)\right\}$$

where the limit of the term inside the first curly bracket on the right hand side is a finite, full column-rank matrix. This limit (multiplied by $I^{*'}$) will act as the scaled Jacobian [see (19) and Remark 11 below].

Definition: For a fixed $d_R \times 1$ vector μ_β, define the local deviation of the null H_0 from the truth as:

$$\sqrt{T}D_{T,R}\Pi'_R(r_0-\beta^0) = \mu_\beta, \tag{14}$$

where $D_{T,R}$ and Π_R are, respectively, $d_R \times d_R$ diagonal and orthogonal matrices dependent on R, Π_{ρ_θ} and D_{T,ρ_θ}. The rather involved definitions of Π_R and $D_{T,R}$ are presented in Appendices A.3.1 and A.3.2.[9]

The intuition behind the matrices Π_R and $D_{T,R}$ is as follows. They are constructed to make

[8] A sketch demonstrating this tradeoff is given in Appendix C.2 by referencing to Lemma 11. Also see footnote 12.

[9] All results go through with a finite μ_β; e.g., (i) $\mu_\beta = 0$ that partially offsets the drawback from footnote 1, (ii) $\mu_\beta \to 0$ as in Remark 18 below. Also note that, the nonstandard nature of (14) is not an artifact of $d_R < d_\theta$, but of rate-entanglement and since we want the results to resemble NM-9.2. For example, suppose instead that one wishes to test $\theta^0 = \theta_0$. So, $d_R = d_\theta$. Then, under the setup of Section 4.1, the proper local deviation should be $\lim_T \operatorname{diag}(1/\delta_{T,1},\ldots,1/\delta_{T,p})B'_T(\theta_0-\theta^0) = \mu_\theta$ for a fixed μ_θ; while under the setup here in Section 4.2, this should resemble (17). Both are still nonstandard.

$D_{T,\rho_\theta}\Pi_{\rho_\theta}R'\Pi_R D_{T,R}$ converge to a finite, full-column rank matrix. Now, writing $V := V(\theta^0)$ and $G_T := E_T[\bar{G}_T(\theta^0)]$ for brevity, and taken together with Remark 10, this implies that $P\left(V^{-1/2}G_T\left(G_T'V^{-1}G_T\right)^{-1}R'\right)$,

$$\text{i.e., } P\left(V^{-1/2}\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}\left(\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}'V^{-1}\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}\right)^{-1}\{D_{T,\rho_\theta}\Pi_{\rho_\theta}R'\Pi_R D_{T,R}\}\right), \quad (15)$$

converges to a finite, idempotent matrix of rank d_R. The importance of this convergence is evident from the second line of the definition of $LM_T(\theta)$ in (4). Finally, again considering the second line of the definition of $LM_T(\theta)$ in (4) and then recalling assumption N and Remark 10, one could write for any θ:

$$\begin{aligned}
& D_{T,R}\Pi_R'R\left(\widehat{G}_T'(\theta)\widehat{V}_T^{-1}\widehat{G}_T(\theta)\right)^{-1}\widehat{G}_T'(\theta)V_T^{-1}\sqrt{T}\bar{g}_T(\theta) \\
= \;& \{D_{T,R}\Pi_R'R\Pi_{\rho_\theta}D_{T,\rho_\theta}\}\left(\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}'V^{-1}\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}\right)^{-1}\{G_T\Pi_{\rho_\theta}D_{T,\rho_\theta}\}'V^{-1}\sqrt{T}\bar{g}_T(\theta^0) \\
& + \sqrt{T}D_{T,R}\Pi_R'R(\theta-\theta^0) + \text{ approximation error.} \qquad (16)
\end{aligned}$$

The definition in (14) characterizes the term $\sqrt{T}D_{T,R}\Pi_R'R(\theta-\theta^0)$ in the last line of (16) as the suitable local deviation that generalizes the classical NM-9.2 representation to our setup [also see footnote 9].

However, as noted in Section 2 (F2), (14) is not sufficient for a study of local efficiency in a rate-entangled setup like ours. More is needed to complete the description of "local". To see this: (i) consider the approximation error in (16); and (ii) note that the definition of $LM_T(\theta)$ in (4) actually involves $\widehat{G}_T(\theta)$ and $\widehat{V}_T(\theta)$ (and not G_T and V), and, therefore, our asymptotic theory can only use (15) with an approximation error. It turns out that, under assumption N, the approximation errors in (i) and (ii) are $o_p(1)$ when θ is such that $\sqrt{T}D_{T,\rho_\theta}^{-1}\Pi_{\rho_\theta}'(\theta-\theta^0) = O_p(1)$. This imposes restrictions on the local deviation in (14) that will turn out to be of fundamental importance in explaining the key feature (F2) related to the local efficiency of the two-step test [see Remark 13 below and Sections 4.2.2-4.2.3].

To impose and/or reflect such restrictions formally, where necessary, we will consider an arbitrary and possibly random sequence $\{\gamma_{S,T} : T \geq 1\} \in \Gamma_S$ such that $\theta_T := A_S^{-1}(r_0', \gamma_{S,T}')'$ satisfies:

$$\sqrt{T}D_{T,\rho_\theta}^{-1}\Pi_{\rho_\theta}'(\theta_T-\theta^0) \equiv \sqrt{T}D_{T,\rho_\theta}^{-1}\Pi_{\rho_\theta}'\left(R_S^1(r_0-\beta^0) + S_S^1(\gamma_{S,T}-\gamma_S^0)\right) = \mu_{T,\theta} \text{ for some } \mu_{T,\theta} = O_p(1). \quad (17)$$

Finally, we define the "effective" linear restrictions for our asymptotic results under the above setup, and the effective scaled Jacobian [also see Remark 10] that will be involved in these results, as follows.

Definition: Define the full row-rank matrix R^* [Appendix A.3.3 contains the details of its construction in (28)] as:

$$R^* := \lim_{T\to\infty} D_{T,R}\Pi_R'R\Pi_{\rho_\theta}D_{T,\rho_\theta} \text{ and note that, } R^*\mu_{T,\theta} \xrightarrow{P} \mu_\beta. \quad (18)$$

Definition: Define the full column-rank matrix G^* [Appendix A.2.3 contains the details of its construction in (25)] as:

$$G^* := \lim_{T\to\infty} E_T[\bar{G}_T(\theta^0)]D_{T,\rho_\theta} \equiv \lim_{T\to\infty} \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\Pi_{\rho_\theta}D_{T,\rho_\theta}. \tag{19}$$

Remark 11: R^* and G^* correspond to R and $G(\theta^0)$ respectively in the classical NM-9.2 setup [see Remark 5]. However, crucially, R and R^* can be very different otherwise [see Remark 16 for an example].

Remark 12: Lastly, we note the modifications to the above definitions that would also make the results in the sequel applicable to the setup of Stock and Wright (2000). To distinguish between the setups, we will use a "tilde" for the corresponding quantities. Resembling N3, define $\tilde{\Lambda}_T := \text{diag}(\tilde{\lambda}_{T,1}1_{\tilde{k}_1}, \ldots, \tilde{\lambda}_{T,\tilde{l}}1_{\tilde{k}_{\tilde{l}}})$ where $\tilde{k}_j > 0$ for $j = 1, \ldots, \tilde{l}$, $\sum_{j=1}^{\tilde{l}} \tilde{k}_j = d_\theta$, $\tilde{\lambda}_{T,j}/\tilde{\lambda}_{T,j+1} = o(1)$ for $j = 1, \ldots, \tilde{l}-1$; $\lim_T \lambda_{T,1} = \infty$ and $\lim_T \tilde{\lambda}_{T,\tilde{l}}/\sqrt{T} < \infty$. Suppose that: $E_T[\bar{G}_T(\theta)] = \tilde{\rho}_{T,\theta}(\theta)\frac{\tilde{\Lambda}_T}{\sqrt{T}}$ where $\tilde{\rho}_{T,\theta}(\theta)$ is some $d_g \times d_\theta$ function with limit $\tilde{\rho}_\theta(\theta)$ uniformly in $\theta \in \Theta$. The crucial difference here is the location of $\tilde{\Lambda}_T$, as opposed to Λ_T earlier, in the matrix-multiplication [c.f. (13)]. Let $\theta = (\theta_1, \ldots, \theta_{d_\theta})'$. If analogous assumptions in N are now maintained in terms of $\tilde{\rho}_\theta(\theta)$ and $\tilde{\Lambda}_T$, then $\tilde{\lambda}_{T,j}$ would be the rate of identification of $(\theta^0_{\tilde{k}_{j-1}+1}, \ldots, \theta^0_{\tilde{k}_j})'$ for $j = 1, \ldots, \tilde{l}$.[10] Hence, the constructions in (14) and (17) can proceed with the corresponding $\Pi_{\rho_\theta}, D_{T,\rho_\theta}, \Pi_R, D_{T,R}$ and R^*, call them $\tilde{\Pi}_{\tilde{\rho}_\theta}, \tilde{D}_{T,\tilde{\rho}_\theta}, \tilde{\Pi}_R, \tilde{D}_{T,R}$ and $\tilde{R}^*$ now, with $\tilde{\Pi}_{\tilde{\rho}_\theta} = I_{d_\theta}$ and $\tilde{D}_{T,\tilde{\rho}_\theta} = \sqrt{T}\tilde{\Lambda}_T^{-1}$ giving $\tilde{\Pi}_R$, $\tilde{D}_{T,R}$ and $\tilde{R}^*$ following Appendices A.3.1-A.3.3. What follows next holds under similar or weaker versions of assumption N; e.g., N7(b) is not needed anymore as $\tilde{\Pi}_{\tilde{\rho}_\theta} = I_{d_\theta}$.

4.2.1 Asymptotic results:

Lemma 5 *Let assumptions O and N hold. For the given S in (3), let r_0 in (14) be such that for the true value γ_S^0 of γ_S, the sequence $\theta_{0,S}^{infs} := A_S^{-1}(r_0', \gamma_S^{0\prime})' \equiv R_S^1 r_0 + S_S^1\gamma_S^0$ satisfies (17). Consider any sequence $\{\theta_T = A_S^{-1}(r_0', \gamma_{S,T}')' : T \geq 1\}$ where $\{\gamma_{S,T} : T \geq 1\}$ is such that (17) holds. Then, the following results hold for $LM_T(\theta_T)$ defined in (4), as $T \to \infty$:*

(a) $LM_T(\theta_T) = LM_T(\theta_{0,S}^{infs}) + o_p(1)$.

(b) $LM_T(\theta_T) \xrightarrow{d} \chi^2_{d_R}$ with non-centrality parameter $\mu_\beta'\left(R^*(G^{*\prime}V^{-1}G^*)^{-1}R^{*\prime}\right)^{-1}\mu_\beta$.

Lemma 6 *Let assumptions O and N hold. For the given S in (3), let r_0 in (14) be such that for the true value γ_S^0 of γ_S, the sequence $\theta_{0,S}^{infs} := A_S^{-1}(r_0', \gamma_S^{0\prime})' \equiv R_S^1 r_0 + S_S^1\gamma_S^0$ satisfies (17). For $\epsilon, \alpha > 0$ such*

[10] A direct generalization of Chaudhuri and Zivot (2011) adhering to Stock and Wright (2000)'s setup would lead to the above structure giving rate-disentangled θ^0. As opposed to (13), here one could, for example, model $E_T[\bar{g}_T(\theta)]$ as [c.f. (13)]:

$$E_T[\bar{g}_T(\theta)] = \sum_{j=1}^{\tilde{l}} \frac{\tilde{\lambda}_{T,j}}{\sqrt{T}}\left(\tilde{\rho}_T^{(j)}(\theta_{\tilde{k}_{j-1}+1}, \ldots, \theta_{d_\theta}) - \tilde{\rho}_T^{(j)}(\theta^0_{\tilde{k}_{j-1}+1}, \ldots, \theta^0_{d_\theta})\right)$$

where $\theta = (\theta_1, \ldots, \theta_{d_\theta})'$, $\tilde{k}_0 = 0$, $\tilde{k}_j > 0$ for $j = 1, \ldots, \tilde{l}$, $\sum_{j=1}^{\tilde{l}} \tilde{k}_j = d_\theta$, and, for $j = 1, \ldots, \tilde{l}$, the functions $\tilde{\rho}_T^{(j)}(.)$ are $d_g \times 1$ deterministic functions satisfying the restrictions in Section 3.3 of Stock and Wright (2000), and other conditions as needed.

that $\epsilon + \alpha < 1$, let $CI_T(\gamma_S;\epsilon)$ be a confidence set for γ_S^0 such that:

$$\sup_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} \sqrt{T} \left\| D_{T,\rho_\theta}^{-1} \Pi_{\rho_\theta}' \left((R_S^1(r_0 - \beta^0) + S_S^1(\gamma_0 - \gamma_S^0) \right) \right\| = O_p(1) \tag{20}$$

where Π_{ρ_θ} and D_{T,ρ_θ} are defined in (22) *and* (23) *in Appendices A.2.1 and A.2.2 respectively. Then,*

$$\inf_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} LM_T \left(A_S^{-1}(r_0', \gamma_0')' \right) = LM_T(\theta_{0,S}^{infs}) + o_p(1) \;\; \textit{for } LM_T(\theta) \textit{ defined in } (4).$$

Remark 13: Lemmas 5 and 6 formalize the discussion of asymptotic equivalence from Remark 5. As we will see in Lemma 7 below, the asymptotic behavior of the two-step test *itself* does not require imposing the condition in (17) since the first step based on $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in (10) automatically incorporates (17) by virtue of (20). Condition (17) is, however, crucial for completely characterizing the local deviation of H_0 for which the invariant-to-S two-step test is asymptotically equivalent to an S-dependent infeasible test in (8).[11] We illustrate analytically and visually in Sections 4.2.2 and 4.2.3 respectively that the characterization of deviations jointly by (14) and (17) becomes important precisely because our setup allows for multiple rates leading to rate-entangled identification of β and γ_S. It would be moot otherwise.

Remark 14: Lemma 5(a) establishes the C-alpha property of $LM_T(\theta)$ that its asymptotic behavior does not depend on certain local deviations of the nuisance parameter γ_S from γ_S^0. This generalizes a key result of Chaudhuri and Zivot (2011) to our setup. Lemma 5(b) describes the asymptotic distribution of $LM_T(\theta)$, and this closely resembles that from the NM-9.2 classical setup; the latter is a special case. Lemma 6 utilizes Lemma 5(a) to establish that the second-step test statistic is asymptotically equivalent to the infeasible statistic under the condition (20) imposed on the first-step confidence set. Then, the asymptotic rejection rate of the improved two-step projection test in (6) follows from Lemma 5(b).

Remark 15: Condition (20) is important for benefitting from the use of (LM) C-alpha, and this is what we ideally expect the first-step confidence set to satisfy. Given the local deviation in (14), it characterizes the worst possible convergence of the first-step confidence set to γ_S^0 that still allows appealing to (17). This helps to establish the local asymptotic efficiency of the two-step test. If $\Lambda_T = \lambda_T I_{d_g}$ for some $\lambda_T \to \infty$ (but $\lim_T \lambda_T/\sqrt{T} < \infty$), i.e., all the rates are equal, then the condition in (20) boils down to $\sup_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} \lambda_T \left\| \gamma_0 - \gamma_S^0 \right\| = O_p(1)$ by virtue of (14). This becomes $\sup_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} \sqrt{T} \left\| \gamma_0 - \gamma_S^0 \right\| = O_p(1)$ if we focus only on the strong identification classical setup as has already done in Chaudhuri and

[11] Recall that the invariance to S for the two-step test in (6) follows once we note (i) and (ii) below. (i) For any S, the optimization problem in (6) is: $\min_{\gamma_S \in \Gamma_S} LM_T(A_S^{-1}(r_0', \gamma_s')')$ such that $h_T(A_S^{-1}(r_0', \gamma_s')') \leq c$ where the inequality constraint represents the first step of the two-step test with $h_T(.)$ as the first-step test statistic and c as the critical value. $h_T(.) = Q_T(.)$ and $c = \chi_{d_g}^2(1-\epsilon)$ in our case [see (10)] since the first step here is inverting the restricted-by-our-H_0 S-test of Stock and Wright (2000). (ii) The minimization problem in (i) is exactly the same as: $\min_{\theta \in \Theta} LM_T(\theta)$ such that $h_T(\theta) \leq c$ and $R\theta = r_0$, where the last equality relates to (2) and (3). The problem in (ii) does not depend on S. Hence, the invariance.

Zivot (2011), Andrews (2017), etc. New insights follow precisely because we allow for multiple rates.

It is, however, clear that our recommended confidence set $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ from Section 4.1 cannot satisfy (20) since $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ can be empty with positive probability. Nevertheless, as noted earlier in footnote 4, we are of the opinion that as long as it satisfies the requirement of Proposition 4, the practical benefit of an empty $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ (with a small ϵ) in: (i) eliminating spurious declines in power and (ii) easing computation, may be worth the cost associated with it. With this caveat of emptiness, which we sidestep by redefining the supremum in (20), we now show that $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ would satisfy (20).

Lemma 7 *Let assumptions O and N hold. Let r_0 satisfy* (14)*. Define the supremum in* (20) *as zero if $CI_T(\gamma_S, \epsilon)$ is empty for a given $\epsilon > 0$. Then, $CI_T(\gamma_S, \epsilon) = CI_T^{SW}(\gamma_S; r_0, \epsilon)$ satisfies* (20) *for $\epsilon > 0$, i.e.,*

$$\sup_{\gamma_0 \in CI_T^{SW}(\gamma_S; r_0, \epsilon)} \sqrt{T} \left\| D_{T,\rho_\theta}^{-1} \Pi_{\rho_\theta}' \left((R_S^1(r_0 - \beta^0) + S_S^1(\gamma_0 - \gamma_S^0)) \right\| = O_p(1).$$

Of course, this is an artificial exercise of sidestepping an empty $CI_T(\gamma_S, \epsilon)$ by redefining the supremum in (20) as zero. In practice, the consequence of an empty $CI_T(\gamma_S, \epsilon)$ is the rejection of H_0 by the two-step test and, therefore, a possible breakdown of its asymptotic equivalence with the infeasible test. As noted above following Chaudhuri and Zivot (2011) and also evident from our simulation results later (Section 4.2.3), this is not necessarily undesirable if ϵ is small. In light of this and also for completeness, we now summarize the takeaways from Lemmas 6 and 7 to present our final asymptotic result in Proposition 8.

Proposition 8 *Let assumptions O and N hold. For the given S in* (3)*, let the sequence of hypothesized value r_0 in* (2) *locally deviate from the truth $\beta^0 := R\theta^0$ following* (14)*, and such that the infeasible sequence $\theta_{0,S}^{infs} := A_S^{-1}(r_0', \gamma_S^{0\prime})' \equiv R_S^1 r_0 + S_S^1 \gamma_S^0$ satisfies* (17)*. Then, for $\epsilon, \alpha > 0$ such that $\epsilon + \alpha < 1$, the asymptotic probability of rejection of r_0 by the improved two-step projection test in* (6)*, based on the choice $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in* (10)*, cannot be smaller than that by the infeasible test in* (8)*.*

4.2.2 Illustration of the asymptotic results with a simple example:

Consider the example from Section 2 (F2) where $d_\theta = 2$, $\theta = (\theta_1, \theta_2)'$ and $R = [1, 1]$. Consider the two choices of S: $S^* = [1, 0]$ and $S^\dagger = [0, 1]$, giving $\gamma_{S^*} = \theta_1$ and $\gamma_{S^\dagger} = \theta_2$. In reference to Remark 13, now we focus on what our local results say about the asymptotic equivalence with the infeasible statistic, in this case for the two choices S^* and $S^\dagger$ and how, through (17), that restricts the local deviation in (14).

Complying with both Antoine and Renault (2012) and Stock and Wright (2000), let, for example,

$$E_T[\bar{G}_T(\theta^0)] = \begin{bmatrix} \frac{\lambda_{T,1}}{\sqrt{T}} \rho_{11} & 0 \\ 0 & \frac{\lambda_{T,2}}{\sqrt{T}} \rho_{22} \end{bmatrix}, \text{ and define } \rho_1 := (\rho_{11}', 0')' \neq 0 \text{ and } \rho_2 := (0', \rho_{22}')' \neq 0. \quad (21)$$

Therefore, $\lambda_{T,1} = \tilde{\lambda}_{T,1}$, $\lambda_{T,2} = \tilde{\lambda}_{T,2}$, $\Pi_{\rho_\theta} = \tilde{\Pi}_{\tilde{\rho}_\theta} = I_2$, $D_{T,\rho_\theta} = \tilde{D}_{T,\tilde{\rho}_\theta} = \sqrt{T}\text{diag}(\lambda_{T,1}^{-1}, \lambda_{T,2}^{-1})$, $\Pi_R = \tilde{\Pi}_R = 1$ and $D_{T,R} = \tilde{D}_{T,R} = \lambda_{T,1}/\sqrt{T}$. This is a setup where identification of θ^0, but not $R\theta^0$, is rate-disentangled.

Remark 16: This setup gives $R^* = \tilde{R}^* = \lim_T[1, \lambda_{T,1}/\lambda_{T,2}]$. Unless $\lambda_{T,1} = \lambda_{T,2}$, i.e., both rates are equal, we have $R^* = \tilde{R}^* = [1, 0] \neq R = [1, 1]$ [see Remark 11]. On the other hand, here $G^* = [\rho_1, \rho_2]$. Therefore, defining $m_{ij} := \rho_i' V^{-1} \rho_j$ for $j = 1, 2$, this implies that the non-centrality parameter in Lemma 5(b) is $\mu_\beta^2(m_{11} - m_{12}^2/m_{22})$, which is what one should obtain in a subvector test for θ_1^0, the element with less strong identification [c.f. Proposition 2.2, Antoine and Renault (2009)]. This is why, for the given R, we referred to R^* earlier (above (18)) as the "effective" linear restrictions for our asymptotic results.

Remark 17: In this case, $\lambda_{T,1}(r_0 - \beta^0) = \mu_\beta$ by (14). However, (17) gives: $\lambda_{T,1}(\theta_{T,1} - \theta_1^0) + \lambda_{T,2}(\theta_{T,2} - \theta_2^0) = \mu_{T,\theta}$, which implies that $\lambda_{T,1}(r_0 - \beta^0) + (\lambda_{T,2} - \lambda_{T,1})(\theta_{T,2} - \theta_2^0) = \mu_{T,\theta}$. Hence, while (14) captures deviations of order up to $\lambda_{T,1}^{-1}$ for $\beta := \theta_1 + \theta_2$, any deviation of order bigger than $\lambda_{T,2}^{-1}$ along the θ_2-axis, that causes this, is beyond the scope of our general results. (On the other hand, things are standard along the θ_1-axis.) This is a consequence of allowing for multiple rates that make $R\theta^0 := \theta_1^0 + \theta_2^0$ rate-entangled.

Remark 18: Remark 17 is not relevant for the asymptotic behavior of the two-step test itself, but is important for its asymptotic equivalence with the infeasible tests. Note that, for the choices $S = S^*, S^\dagger$, we have $\theta_{0,S}^{infs}$ as $\theta_{0,S^*}^{infs} = (\gamma_{S^*}^0 := \theta_1^0, r_0 - \gamma_{S^*}^0)'$ and $\theta_{0,S^\dagger}^{infs} = (r_0 - \gamma_{S^\dagger}^0, \gamma_{S^\dagger}^0 := \theta_2^0)'$ respectively. Consider r_0 satisfying (14), i.e., $(r_0 - \beta^0) = \mu_\beta/\lambda_{T,1}$ for some constant $\mu_\beta \neq 0$, which, for θ_{0,S^*}^{infs} and $\theta_{0,S^\dagger}^{infs}$ respectively, means that $(\theta_{T,2} - \theta_2^0) = \mu_\beta/\lambda_{T,1}$ and $(\theta_{T,1} - \theta_1^0) = \mu_\beta/\lambda_{T,1}$. Hence, from Remark 17, we know that for a constant $\mu_\beta \neq 0$ in (14), $\theta_{0,S^\dagger}^{infs}$ falls directly under the scope of our general results, while θ_{0,S^*}^{infs} does not. For the latter, we need $\mu_\beta = O(\lambda_{T,1}/\lambda_{T,2})$, a deviation more local than what is implied by (14) *alone*.

Remark 19: This is the role of (17). It refines the deviation in (14) to precisely tell where the asymptotic equivalence holds. Interestingly, since the two-step test (and the standard plug-in tests) is invariant to S, this means that irrespective of what it uses for S, the asymptotic equivalence holds in a larger region with the infeasible test that uses $S = S^\dagger$, i.e., the the better identified nuisance parameter $\gamma_S = \theta_2$. This nicely aligns with the practical implementation of the two-step projection test since it is generally computationally easiest to use the same, i.e., the most strongly identified γ_S, if possible [see Remark 1].

4.2.3 Simulation study with a linear instrumental variables model:

Let us now visually demonstrate the points made in Section 4.2.2 with the help of a linear instrumental variables model, the leading example of GMM. A linear (in θ) $g(Z_t; \theta)$ gives the usual simplifications.[12]

[12] For example, the rate-restrictions $\lambda_{T,j}(\theta_{T,j} - \theta_j^0) = O_p(1)$ for $j = 1, 2$ in (17) can be weakened for the properly scaled terms inside the projection matrix $P(.)$ in $LM_T(\theta)$ in (4) to converge to the concerned limits. All we need for this convergence are $(\theta_{T,1} - \theta_1^0) = o_p(1)$ and $\lambda_{T,2}(\theta_{T,2} - \theta_2^0) = o_p(\lambda_{T,1})$ [c.f. (16) and the discussion around it]. To see this, let $\theta_T \xrightarrow{P} \theta^0$. Then, Lemma 11 (a), (b) and (c) hold [see Appendix C.1]. On the other hand, for $j = 1, 2$, use N2 to obtain that $\frac{\sqrt{T}}{\lambda_{T,j}} \bar{g}_T(\theta_T) = O_p\left(\frac{1}{\lambda_{T,j}}\right) + \frac{\lambda_{T,1}}{\lambda_{T,j}} \rho_1(\theta_{T,1} - \theta_1^0) + \frac{\lambda_{T,2}}{\lambda_{T,j}} \rho_2(\theta_{T,2} - \theta_2^0)$, which is $o_p(1)$ if, additionally, $\frac{\lambda_{T,2}}{\lambda_{T,j}}(\theta_{T,2} - \theta_2^0) = o_p(1)$,

Simulation design: Consider an i.i.d. sample $\{Z_{t,T}\}_{t=1}^{T}$ written as $\{Z_t := (y_t, X_{1t}, X_{2t}, W_t')'\}_{t=1}^{T}$ where

$$\begin{aligned} \text{dependent variable: } y_t &= X_{1t}\theta_1^0 + X_2\theta_2^0 + u_t, \\ \text{endogenous regressors: } X_{jt} &= W_t'\pi_{jT} + v_{jt} \text{ for } j = 1, 2, \end{aligned}$$

while the instruments $W_t \sim N(0, I_4)$ are independent of the model errors (u_t, v_{1t}, v_{2t}): $u_t \sim N(0,1)$, $v_{jt} \sim N(0,1)$ with $Cov(u_t, v_{jt}) = .8$ and $Cov(v_{1t}, v_{2t}) = .3$ for $j = 1, 2$. We take $\theta_1^0 = \theta_2^0 = .5$.

The moment vector corresponding to (1) is $g(Z_t;\theta) = W_t(y_t - X_{1t}\theta_1 - X_2\theta_2)$, which is 4×1 dimensional.

Note that, this design closely follows Chaudhuri and Zivot (2011) except that we will now simplify the structure of π_{1T} and π_{2T} such that it conforms not only to the setup of Stock and Wright (2000) but also to that of Antoine and Renault (2012). In particular, relating to the setup and discussion in Section 4.2.2 and using the same scale-factors $\sqrt{4/3}$ and 20 as in Chaudhuri and Zivot (2011) to further contrast the nonclassical and classical rates, we specify π_{jT} for $j = 1, 2$ such that (21) holds with:

- $\rho_1 = \sqrt{4/3}(1,1,0,0)'$ if $\lambda_{T,1} \neq \sqrt{T}$, and $\rho_1 = 20(1,1,0,0)'$ if $\lambda_{T,1} = \sqrt{T}$ (i.e., classical rate),
- $\rho_2 = \sqrt{4/3}(0,0,1,1)'$ if $\lambda_{T,2} \neq \sqrt{T}$, and $\rho_2 = 20(0,0,1,1)'$ if $\lambda_{T,2} = \sqrt{T}$ (i.e., classical rate).

We take $T = 100$ and consider six different specifications for the identification rate/strength of the elements of θ^0: (i) $\lambda_{T,1} = \lambda_{T,2} = 1$, (ii) $\lambda_{T,1} = 1, \lambda_{T,2} = T^{1/6}$, (iii) $\lambda_{T,1} = 1, \lambda_{T,2} = \sqrt{T}$, (iv) $\lambda_{T,1} = T^{1/6}, \lambda_{T,2} = T^{1/6}$, (v) $\lambda_{T,1} = T^{1/6}, \lambda_{T,2} = \sqrt{T}$, and (vi) $\lambda_{T,1} = \sqrt{T}, \lambda_{T,2} = \sqrt{T}$.

Only (vi), i.e., the classical scenario (a special case of our setup), has so far been considered in the related literature on efficiency. As noted before, (iv)-(v) are covered by our setup in Section 4.2 in spite of violating N7(b) since $g(Z_t;\theta)$ is linear in θ. Efficiency properties under (i)-(iii) are out of the scope of Section 4.2. ((i)-(vi) are all under the scope of Section 4.1 if instead interest lies in the empirical size.)

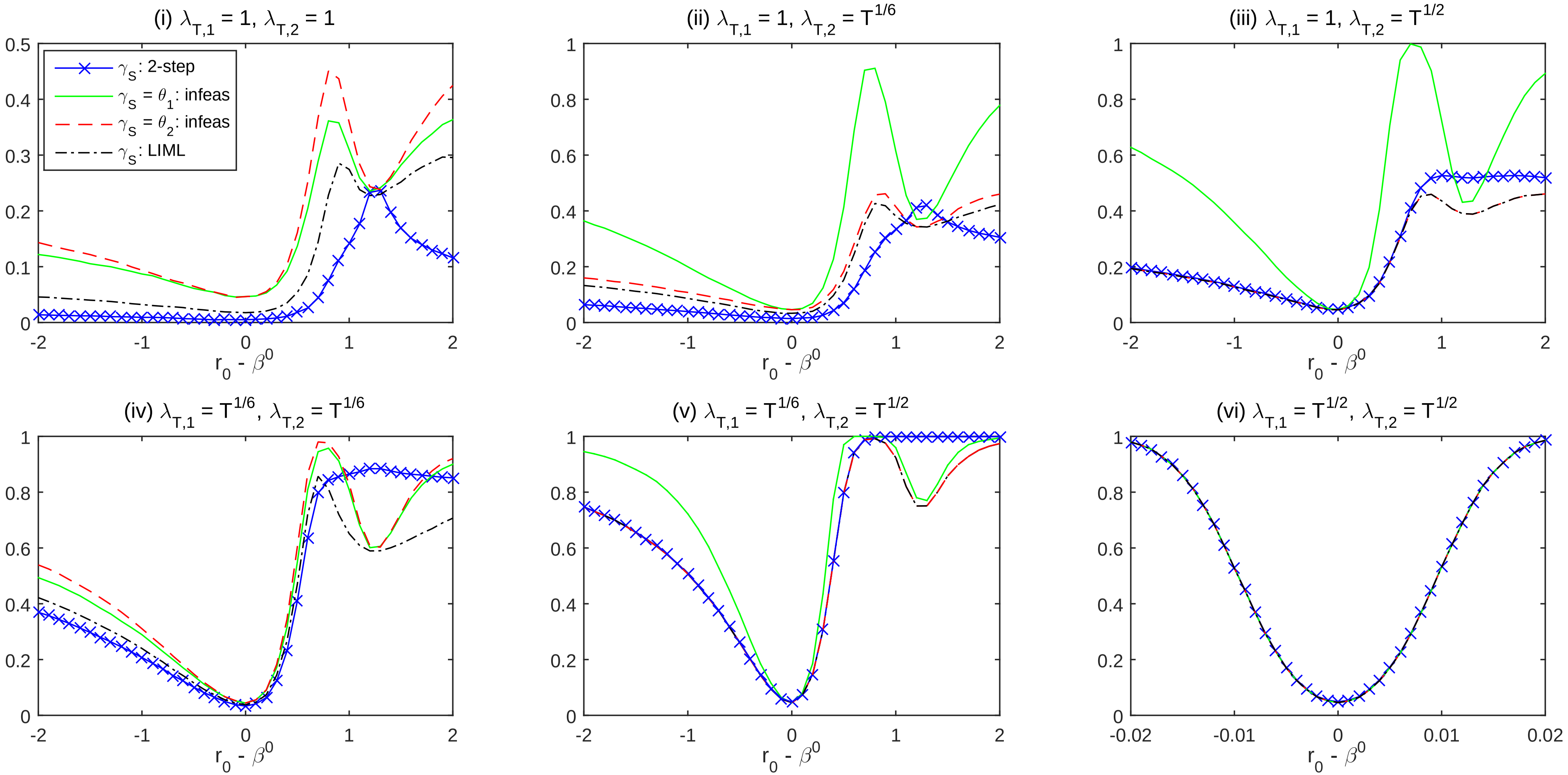
We take $R = [1, 1]$ in (2). We consider two choices of S in (3): $S^* = [1, 0]$ and $S^\dagger = [0, 1]$ giving the true γ_S as $\gamma_{S^*}^0 = \theta_1^0$ and $\gamma_{S^\dagger}^0 = \theta_2^0$. We consider six different tests, three for each choice of S:

- The infeasible test in (8).
- The standard plug-in test that rejects H_0 if $LM_T(r_0, \widehat{\gamma_S}(r_0)) > \chi^2_{d_R}(1-\alpha)$ where $\widehat{\gamma_S}(r_0)$ is the restricted-by-H_0 CU-GMM estimator of γ_S, i.e., $\widehat{\gamma_S}(r_0) := \arg\min_{\gamma\in\Gamma_S} Q_T(A_S^{-1}(r_0', \gamma')')$ with $Q_T(.)$ as in (11). Here, $\widehat{\gamma_S}(r_0)$ is the restricted limited information maximum likelihood (LIML) estimator.
- The two-step projection test in (6) with $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in (10) as the first-step confidence set.

Simulation results: Figure 1 plots the empirical rejection probability of these tests with $S = S^*, S^\dagger$ and under specifications (i)-(vi). Since the standard plug-in test and the two-step projection test are both

a less stringent rate restriction than (17). It is also worth noting here that since identification is better than weak, i.e., $\lambda_{T,1} \to \infty$, when taken in conjunction with Lemma 7, this discussion explains the part in Remark 9 that led to footnote 8.

Figure 1: Empirical rejection probabilities of the two-step projection test (2-step) in (6) with $\epsilon = .005, \alpha = .045$, the infeasible test (infeas) in (8) with $\alpha = .045$, and the standard plug-in test (LIML) based on the restricted-by-H_0 CU-GMM (LIML) estimator for γ_S, with $\alpha = .045$. Two choices of γ_S, i.e., $\gamma_S = \theta_1$ and $\gamma_S = \theta_2$, are employed for the infeasible test. The other tests are invariant to S. Results are based on 10,000 Monte Carlo trials. Horizontal axis: deviation of H_0 in (2) from the truth [see (14)]. Title: Identification strength that corresponds to specifications (i)-(vi) respectively.



invariant to the choice of S, only one plot for each test is reported.[13] Results under different variations of our design and rate-specifications are qualitatively similar, and are available from the author.

We find that the simulation results under (v) and (vi) and, to a slightly lesser extent, under (iv) corroborate our asymptotic efficiency results even for $T = 100$. The results are also encouraging showing competitive power for the two-step test under (ii) and (iii) that are actually outside our scope.

Crucially, when $\lambda_{T,1} \neq \lambda_{T,2}$, the two infeasible tests behave very differently as H_0 deviates a bit far from the truth. Then, we find that the two-step and the standard plug-in tests, both of which are invariant to S, resemble that infeasible test which uses $\gamma_S = \theta_2$, the better identified component of θ^0.[14] In other words, we observe that the rejection rate of a feasible invariant test, i.e., the two-step projection test or the standard plug-in test, can only be as large as that of the less powerful infeasible test. This observation is intuitive since the more powerful infeasible tests assume knowledge of the less strongly identified nuisance parameters and, therefore, it is impossible for a feasible test (that cannot make such assumptions) to resemble the behavior of such infeasible tests in a meaningfully large region/interval.

To relate this observation to our general results, recall that, while we established the asymptotic equivalence of the invariant two-step tests with all the non-invariant infeasible tests when deviations satisfied (14), the condition (17) put restrictions on the deviations depending on the specific S-dependent infeasible test under consideration. The consequences of such restrictions were prominently reflected in our simulation study. Note that, they mattered only when we considered multiple rates leading to rate-entangled identification, but were moot otherwise when we considered a single rate as in the related literature on efficiency, e.g., Chaudhuri and Zivot (2011), Andrews (2017). Thus, they confirm the novel aspect of our local efficiency result, i.e., the key feature (F2), that we sought to highlight in our paper.

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[13]The plug-in test is not invariant to S if H_0 is not imposed while estimating γ_S. This has adverse consequences on size [see Appendix D] under specifications where the standard plug-in test works [Theorem 6, Guggenberger and Smith (2005)].

[14]More extensive simulations (1 million trials and grid size .001) not reported here suggest that when $\lambda_{T,1} \neq \lambda_{T,2}$, the two infeasible tests behave similarly roughly in the interval $[-.1, .1]$ for $r_0 - \beta^0$. On the other hand, their difference under (i) and (iv), where $\lambda_{T,1} = \lambda_{T,2}$, is due to small sample size; and it vanishes if, e.g., $T = 1000$ for (i) and $T = 250$ for (iv).

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Appendix A: Important constructions and definitions for Section 4

A.1 UBT and LBT Constructions:

We extensively use the following constructions that are adapted from the original work of Antoine and Renault (2012), Andrews and Cheng (2014), Cheng (2015), etc. Let $\{W_T = [W_{T,1}, \dots, W_{T,m_T}] : T \geq 1\}$ be a sequence of $r \times c$ (for some r, c) matrix of full row-rank $r(\leq c)$ where $W_{T,j}$ is $r \times c_{T,j}$ (and empty if $c_{T,j} = 0$) for $j = 1, \dots, m_T$ and such that $\sum_{j=1}^{m_T} c_{T,j} = c$ for each $T \geq 1$.

A.1.1 UBT-Construction: An upper block-triangular (UBT) construction

We construct a sequence of $r \times r$ matrix $\{\Pi_T = [\Pi_{T,1}, \dots, \Pi_{T,m_T}] : T \geq 1\}$ such that the $c \times r$ matrix $W_T'\Pi_T$ has an UBT structure for each $T \geq 1$. For any given T, the following steps give such a Π_T.

- Let $\text{rank}(W_{T,m_T}) = c^*_{T,m_T} \leq \min(r, c_{m_T})$. Define Π_{T,m_T} as the $r \times c^*_{T,m}$ matrix such that its columns form an orthogonal basis for the column space of W'_{T,m_T}. Stop if $m_T = 1$.
- Let $\text{rank}([W_{T,m_T-1}, W_{T,m_T}]) - \text{rank}(W_{T,m_T}) = c^*_{T,m_T-1} \leq \min(r, c_{m_T-1})$. Define Π_{T,m_T-1} as the $r \times c^*_{T,m_T-1}$ matrix such that the columns of $[\Pi_{T,m_T-1}, \Pi_{T,m_T}]$ form an orthogonal basis for the column space of $[W_{T,m_T-1}, W_{T,m_T}]'$. Stop if $m_T = 2$.
- Continue step-by-step, as above, for $j = m_T - 2, \dots, 1$ and for each j, define $\Pi_{T,j}$ as the $r \times c^*_{T,j}$ matrix, where $c^*_{T,j} = \text{rank}([W_{T,j}, \dots, W_{T,m_T}]) - \text{rank}([W_{T,j+1}, \dots, W_{T,m_T}) \leq \min(r, c_{T,j})$, such that the columns of $[\Pi_{T,j}, \dots, \Pi_{T,m_T}]$ form an orthogonal basis for the column space of $[W_{T,j}, \dots, W_{T,m_T}]'$.

As a convention, $\Pi_{T,j}$ is an empty matrix if $c^*_{T,j} = 0$. Π_T is an orthogonal matrix by construction and

(i) for some integer $q_T \in \{1, \dots, \min(r, m_T)\}$, the q_T blocks $W'_{T,j_{k,T}}\Pi_{T,j_{k,T}}$ for $k = 1, \dots, q_T$, and where $1 \leq j_{1,T} < \dots < j_{q_T,T} \leq m_T$, each has full column-rank $c^*_{T,j_{k,T}} > 0$ satisfying $\sum_{k=1}^{q_T} c^*_{T,j_{k,T}} = r$;

(ii) $W'_{T,j}\Pi_{T,k} = 0$, a zero matrix of suitable (according to the above) dimension, for all $1 \leq k < j \leq m_T$.

A.1.2 LBT-Construction: A lower block-triangular (LBT) construction

We construct a sequence of $r \times r$ matrix $\{\Pi_T = [\Pi_{T,1}, \dots, \Pi_{T,m_T}] : T \geq 1\}$ such that the $c \times r$ matrix $W_T'\Pi_T$ has a BLT structure for each $T \geq 1$. For any given T, the following steps give such a Π_T. (This is same as the UBT-Construction, but in reverse order. Hence to save new notation, we continue to use the same notation as in the UBT-Construction and hope that this is not confusing.)

- Let $\text{rank}(W_{T,1}) = c^*_{T,1} \leq \min(r, c_1)$. Define $\Pi_{T,1}$ as the $r \times c^*_{T,1}$ matrix such that its columns form an orthogonal basis for the column space of $W'_{T,1}$. Stop if $m_T = 1$.
- Let $\text{rank}([W_{T,1}, W_{T,2}]) - \text{rank}(W_{T,1}) = c^*_{T,2} \leq \min(r, c_2)$. Define $\Pi_{T,2}$ as the $c \times c^*_{T,2}$ matrix such that the columns of $[\Pi_{T,1}, \Pi_{T,2}]$ form an orthogonal basis for the column space of $[W_{T,1}, W_{T,2}]'$. Stop if $m_T = 2$.

- Continue step-by-step, as above, for $j = 3, \ldots, m_T$ and for each j, define $\Pi_{T,j}$ as the $r \times c^*_{T,j}$ matrix, where $c^*_{T,j} = \text{rank}([W_{T,1}, \ldots, W_{T,j}]) - \text{rank}([W_{T,1}, \ldots, W_{T,j-1}) \leq \min(r, c_{T,j})$, such that the columns of $[\Pi_{T,1}, \ldots, \Pi_{T,j}]$ form an orthogonal basis for the column space of $[W_{T,1}, \ldots, W_{T,j}]'$.

As a convention, $\Pi_{T,j}$ is an empty matrix if $c^*_{T,j} = 0$. Π_T is an orthogonal matrix by construction and

(i) for some integer $q_T \in \{1, \ldots, \min(r, m_T)\}$, the q_T blocks $W'_{T,j_{k,T}}\Pi_{T,j_{k,T}}$ for $k = 1, \ldots, q_T$, and where $1 \leq j_{1,T} < \ldots < j_{q_T,T} \leq m_T$, each has full column-rank $c^*_{T,j_{k,T}} > 0$ satisfying $\sum_{k=1}^{q_T} c^*_{T,j_{k,T}} = r$;

(ii) $W'_{T,j}\Pi_{T,k} = 0$, a zero matrix of suitable (according to the above) dimension, for all $1 \leq j < k \leq m_T$.

A.2 Construction of Π_{ρ_θ}, D_{T,ρ_θ} and G^*:

The efficient rate-disentangled directions of θ that are identified from (13) under our assumptions are given by $\Pi^{-1}_{\rho_\theta}\theta$ where Π_{ρ_θ} is a $d_\theta \times d_\theta$ orthogonal matrix, and the appropriate rates along these directions, in the given order, are given by the $d_\theta \times d_\theta$ diagonal matrix $\sqrt{T}D^{-1}_{T,\rho_\theta}$ [see Antoine and Renault (2012)].

A.2.1 Construction of Π_{ρ_θ}:

Let $\rho_\theta := \rho_\theta(\theta^0)$, i.e., $\partial\rho(\theta^0)/\partial\theta'$. Using N3 write $I^*\Lambda_T\rho_\theta \equiv I^*\Lambda_T I^{*'} I^*\rho_\theta = \left[\lambda_{T,1}\rho'_{\theta,1}, \ldots, \lambda_{T,l}\rho'_{\theta,l}\right]'$ where $\rho_{\theta,j}(\theta)$ is a $k_j \times d_\theta$ matrix for $j = 1, \ldots, l$. Take $W_T = \left[\rho'_{\theta,1}, \ldots, \rho'_{\theta,l}\right] = (I^*\rho_\theta)'$ (not depending on T) in the UBT-Construction in Appendix A.1.1. To emphasize the non-dependence on T, write W_T as W, and accordingly write the rest of the notation from the UBT-Construction. Thus $r = d_\theta$, $c = d_g$ and $m = l$ in terms of the notation from the UBT-Construction. W is full row-rank r $(= d_\theta)$ by N4.

$\Pi_{\rho_\theta} = [\Pi_{\rho_\theta,1}, \ldots, \Pi_{\rho_\theta,l}]$ is the $d_\theta \times d_\theta$ matrix Π from the UBT-Construction with $W = \left(I^*\rho_\theta(\theta^0)\right)'$. (22)

A.2.2 Construction of D_{T,ρ_θ}:

The construction of D_{T,ρ_θ} depends on the matrix $I^*\Lambda_T I^{*'} I^*\rho_\theta(\theta^0)\Pi_{\rho_\theta}$. Let $c^*_{\rho_\theta,j} = c^*_j \geq 0$ denote the number of columns of $\Pi_{\rho_\theta,j}$ for $j = 1, \ldots, l$, and $q_{\rho_\theta} = q$ from (i) in the UBT-Construction (of Π_{ρ_θ}). Let $(j_1, \ldots, j_{q_{\rho_\theta}})$ denote the indices such that the block $\rho_{\theta,j_i}\Pi_{\rho_\theta,j_i}$ of dimension $k_{j_i} \times c^*_{\rho_\theta,j_i}$ is full column-rank $c^*_{\rho_\theta,j_i} > 0$ for $i = 1, \ldots, q_{\rho_\theta}$ and $\sum_{i=1}^{q_{\rho_\theta}} c^*_{\rho_\theta,j_i} = d_\theta$. Thus, the corresponding block of $I^*\Lambda_T I^{*'} I^*\rho_\theta(\theta^0)\Pi_{\rho_\theta}$ is $\lambda_{T,j_i}\rho_{\theta,j_i}\Pi_{\rho_\theta,j_i}$. Accordingly, for $I^*\Lambda_T I^{*'} I^*\rho_\theta(\theta^0)\Pi_{\rho_\theta}$, the columns from $(d_\theta - \sum_{i'=i}^{q_{\rho_\theta}} c^*_{\rho_\theta,j_{i'}})$ to $(d_\theta - \sum_{i'=i}^{q_{\rho_\theta}} c^*_{\rho_\theta,j_{i'}} + c^*_{\rho_\theta,j_i})$ for $i = 1, \ldots, q_{\rho_\theta}$ are represented by the $d_g \times c^*_{\rho_\theta,j_i}$ matrix:

$$\left[\lambda_{T,1}(\rho_{\theta,1}\Pi_{\rho_\theta,1})', 0'\right]' \text{ if } j_i = 1,$$

$$\left[\lambda_{T,1}(\rho_{\theta,1}\Pi_{\rho_\theta,j_i})', \ldots, \lambda_{T,j_i}(\rho_{\theta,j_i}\Pi_{\rho_\theta,j_i})', 0'\right]' \text{ otherwise.}$$

In both cases: $j_i = 1$ and $j_1 > 1$, the 0's inside the big matrices denote sub-matrices of zeros with number of rows, which can be zero, such that the number of rows of the corresponding big matrix is d_g.

Now, conforming to this above structure, define the $d_\theta \times d_\theta$ matrix D_{T,ρ_θ} as:

$$D_{T,\rho_\theta} := \sqrt{T}\text{diag}\left(\lambda_{T,j_1}^{-1} 1_{c^*_{\rho_\theta,j_1}}, \ldots, \lambda_{T,j_{q_{\rho_\theta}}}^{-1} 1_{c^*_{\rho_\theta,j_{q_{\rho_\theta}}}}\right). \tag{23}$$

A.2.3 Construction of G^*:

Define the $d_g \times d_\theta$ matrix $G^\dagger$ as the following limit:

$$G^\dagger := \lim_{T\to\infty} \frac{1}{\sqrt{T}} I^* \Lambda_T I^{*'} I^* \rho_\theta(\theta^0) \Pi_{\rho_\theta} D_{T,\rho_\theta}. \tag{24}$$

By construction, $G^\dagger$ is finite, and its columns from $(d_\theta - \sum_{i'=i}^{q_{\rho_\theta}} c^*_{\rho_\theta,j_{i'}})$ to $(d_\theta - \sum_{i'=i}^{q_{\rho_\theta}} c^*_{\rho_\theta,j_{i'}} + c^*_{\rho_\theta,j_i})$ for $i = 1, \ldots, q_{\rho_\theta}$ are represented by the $d_g \times c^*_{\rho_\theta,j_i}$ matrix:

$$\begin{aligned} &\left[(\rho_{\theta,1}\Pi_{\rho_\theta,1})', 0'\right]' \;\; j_i = 1, \\ &\left[0', (\rho_{\theta,j_i}\Pi_{\rho_\theta,j_i})', 0'\right]' \;\; \text{otherwise.} \end{aligned}$$

(As above, 0 denotes sub-matrices of zeros with number of rows, which can be zero, such that the number of rows of the corresponding matrix is d_θ.) Naturally, under our assumptions $G^\dagger$ is full column-rank.

Now define the $d_g \times d_\theta$ finite matrix of full column-rank G^* as:

$$G^* := I^{*'} G^\dagger. \tag{25}$$

A.3 Construction of Π_R, $D_{T,R}$ and R^*:

Π_R and $D_{T,R}$ are quantities used to characterize the appropriate local deviation of the null from the truth. Their construction depends on the constructions of Π_{ρ_θ} and D_{T,ρ_θ}.

A.3.1 Construction of Π_R:

Take $W_T = R\Pi_{\rho_\theta} = [W_{T,1} = R\Pi_{\rho_\theta,j_1}, \ldots, W_{T,q_{\rho_\theta}} = R\Pi_{\rho_\theta,j_{\rho_\theta}}]$ (not depending on T) in the LBT-Construction in Appendix A.1.2. Note that the partition of W_T was informed by the indices $j_1, \ldots, j_{q_{\rho_\theta}}$ defined immediately after constructing Π_{ρ_θ}. These indices do not depend on T, and they also informed the construction of D_{T,ρ_θ}. Once again, to emphasize the non-dependence on T, write W_T as W, and accordingly write the rest of the notation from the LBT-Construction. Thus $r = d_R$, $c = d_\theta$ and $m = q_{\rho_\theta}$. W is full row-rank by the definition of R, Π_{ρ_θ} and Lemma 10 (in Appendix C).

$$\Pi_R = [\Pi_{R,1}, \ldots, \Pi_{R,q_{\rho_\theta}}] \text{ is the } d_R \times d_R \text{ matrix } \Pi_T \text{ from the LBT-Construction with } W = R\Pi_{\rho_\theta}. \tag{26}$$

A.3.2 Construction of $D_{T,R}$:

The construction of $D_{T,R}$ depends on the matrix $D_{T,\rho_\theta}\Pi'_{\rho_\theta}R'\Pi_R$. Let $c^*_{R,j} = c^*_j \geq 0$ denote the number of columns of $\Pi_{R,j}$ for $j = 1,\ldots,q_{\rho_\theta}$, and $q_R = q$ from (i) in the LBT-Construction (of Π_R). Let $(j_{n_1},\ldots,j_{n_{q_R}})$ denote the sub-indices of the indices $(j_1,\ldots,j_{q_{\rho_\theta}})$ such that the block $\Pi'_{\rho_\theta,j_{n_i}}R'\Pi_{R,n_i}$ of dimension $c^*_{\rho_\theta,j_{n_i}} \times c^*_{R,n_i}$ is full column-rank $c^*_{R,n_i} > 0$ for $i = 1,\ldots,q_R$ and $\sum_{i=1}^{q_R} c^*_{R,n_i} = d_R$. Thus, the corresponding block of $D_{T,\rho_\theta}\Pi'_{\rho_\theta}R'\Pi_R$ is $\frac{\sqrt{T}}{\lambda_{T,j_{n_i}}}\Pi'_{\rho_\theta,j_{n_i}}R'\Pi_{R,n_i}$. Accordingly, for $D_{T,\rho_\theta}\Pi'_{\rho_\theta}R'\Pi_R$, the columns from $(d_R - \sum_{i'=i}^{q_R} c^*_{R,n_{i'}})$ to $(d_R - \sum_{i'=i}^{q_R} c^*_{R,n_{i'}} + c^*_{R,n_i})$ for $i = 1,\ldots,q_R$ are represented by the $d_\theta \times c^*_{R,n_i}$ matrix:

$$\begin{aligned}
&\left[0', \frac{\sqrt{T}}{\lambda_{T,j_{q_{\rho_\theta}}}}\left(\Pi'_{\rho_\theta,j_{q_{\rho_\theta}}}R'\Pi_{R,q_{\rho_\theta}}\right)'\right]' \text{ if } n_i = q_{\rho_\theta}, \\
&\left[0', \frac{\sqrt{T}}{\lambda_{T,j_{n_i}}}\left(\Pi'_{\rho_\theta,j_{n_i}}R'\Pi_{R,n_i}\right)', \ldots, \frac{\sqrt{T}}{\lambda_{T,j_{q_{\rho_\theta}}}}\left(\Pi'_{\rho_\theta,j_{q_{\rho_\theta}}}R'\Pi_{R,n_i}\right)'\right]' \text{ otherwise.}
\end{aligned}$$

In both cases, 0 represents the sub-matrix of zeros with number of rows that make the number of rows of the corresponding matrix equal to d_θ.

Now, conforming to this above structure, define the $d_R \times d_R$ matrix $D_{T,R}$ as:

$$D_{T,R} := T^{-1/2}\mathrm{diag}\left(\lambda_{T,j_{n_1}}1_{c^*_{R,n_1}},\ldots,\lambda_{T,j_{n_{q_R}}}1_{c^*_{R,n_{q_R}}}\right). \tag{27}$$

A.3.3 Construction of R^*:

Define the $d_R \times d_\theta$ matrix R^* as the transpose of the following limit:

$$R^{*'} := \lim_{T\to\infty} D_{T,\rho_\theta}\Pi'_{\rho_\theta}R'\Pi_R D_{T,R}. \tag{28}$$

By construction, $R^{*'}$ is finite, and its columns from $(d_R - \sum_{i'=i}^{q_R} c^*_{R,n_{i'}})$ to $(d_R - \sum_{i'=i}^{q_R} c^*_{R,n_{i'}} + c^*_{R,n_i})$ for $i = 1,\ldots,q_R$ are represented by the $d_\theta \times c^*_{R,n_i}$ matrix:

$$\begin{aligned}
&\left[0', \left(\Pi'_{\rho_\theta,j_{q_{\rho_\theta}}}R'\Pi_{R,q_{\rho_\theta}}\right)'\right]' \text{ if } n_i = q_{\rho_\theta}, \\
&\left[0', \left(\Pi'_{\rho_\theta,j_{n_i}}R'\Pi_{R,n_i}\right)', 0'\right]' \text{ otherwise.}
\end{aligned}$$

(As above, 0 denotes sub-matrices of zeros with number of rows, which can be zero, such that the number of rows of the corresponding matrix is d_θ). Naturally, under our assumptions R^* is full row-rank.

Supplemental Appendix B: For the references from Section 3

Appendix B.1: Efficient influence function for $\beta^0 := R\theta^0$ under (1)

It is well-known that under the assumptions that (1) holds, $G(\theta^0)$ is full column-rank, and $V(\theta^0)$ is positive definite: the efficient estimator of $R\theta^0$ has an asymptotically linear representation $-\sqrt{T}l_T(\theta^0)+o_p(1)$. Unfortunately, we could not find a paper to cite the proof of it. So we provide a standard proof.

Lemma 9 *Let $\{Z_t\}_{t=1}^T$ be i.i.d. copies of a random variable Z, and let (1) holds. If $G := \frac{\partial}{\partial\theta'}E[g(Z;\theta)]_{\theta=\theta^0}$ is a full column-rank $d_g \times d_\theta$ matrix and $V := E[g(Z;\theta^0)g'(Z;\theta^0)]$ is a $d_g \times d_g$ positive definite matrix, then the asymptotic variance lower bound for any regular estimator of the $d_R \times 1$ parameter vector $\beta^0 := R\theta^0$ where $d_R \leq d_\theta$ is $(R(G'V^{-1}G)R')^{-1}$. The regular estimator whose asymptotic variance attains this bound has the asymptotically linear representation $\sqrt{T}(\widehat{\beta^0}-\beta^0) = -\sqrt{T}l_T(\theta^0)+o_p(1)$.*

Proof: Consider a parametric path ξ of the distribution of Z such that for the unique value ξ^0 we have the joint density $f_{\xi^0}(z) = f(z)$, the true density. Let $s_\xi(Z)$ denote the score with respect to ξ. Without any other restrictions, the tangent space for the model is simply $\mathcal{T} = a(z)$ where $a(z)$ satisfies $E[a(Z)] = 0$, and $E[.]$ equivalently stands for $E_{\xi^0}[.]$. Since $d_g > d_R$, (1) equivalently requires that for any given $d_R \times d_g$ matrix B, the relation $BE[g(Z;\theta^0)] = 0$ holds. Take B as full row-rank without loss of generality. Now, differentiating with respect to ξ under the expectation we obtain $\frac{\partial\theta(\xi^0)}{\partial\xi} = -(BG)^{-1}E[Bg(Z;\theta^0)s_{\xi^0}(Z)]$ and thus $\frac{\partial\beta^0(\xi^0)}{\partial\xi} = -R(BG)^{-1}E[Bg(Z;\theta^0)s_{\xi^0}(Z)]$. Therefore, any regular estimator for β^0 will be asymptotically linear with the influence function $\varphi(B) := -R(BG)^{-1}Bg(Z;\theta^0)$. Given the structure of the tangent space $\mathcal{T}$, (1) implies that the projection of this influence function $\varphi(B)$ onto $\mathcal{T}$ is $\varphi(B)$ itself. For this given B, $Var(\varphi(B)) = \Sigma(B) := R(BG)^{-1}BVB'(BG)^{-1'}R'$. Thus the efficient influence function is obtained by choosing $B^* := \arg\min_B \Sigma(B) = G'V^{-1}$, giving $\Sigma(B^*) = R(G'V^{-1}G)^{-1}R'$ and $\varphi(B^*) = -R(G'V^{-1}G)^{-1}G'V^{-1}g(Z;\theta^0)$. This completes the proof. ■

Appendix B.2: The second-step test statistic $LM_{T,S}^{\mathbf{ES}}(\theta)$ in Chaudhuri and Zivot (2011):

Given the choice of S in (3), the scores for β and γ_s, by which we mean here the population version of the optimal rotations, in the efficient GMM sense, of $\bar{g}_T(\theta^0)$ along the directions of β and γ_s are:

$$l_{\beta,S,T}(\theta) := R_S^{1'}G'(\theta)V^{-1}(\theta)\bar{g}_T(\theta) \quad \text{and} \quad l_{\gamma_S,S,T}(\theta) := S_S^{1'}G'(\theta)V^{-1}(\theta)\bar{g}_T(\theta)$$

respectively. It is important to note that while the definition of $\beta := R\theta$ does not depend on S, the score for β in the re-parameterized model generally depends on S through R_S^1 [see Remark 20].

Following Chaudhuri and Zivot (2011), the efficient score for β would be the residual from a regression:

$$l_{\beta.\gamma_S,S,T}(\theta) := l_{\beta,S,T}(\theta) - \mathrm{Cov}\left(\sqrt{T}l_{\beta,S,T}(\theta), \sqrt{T}l_{\gamma_S,S,T}(\theta)\right) \mathrm{Var}^{-1}\left(\sqrt{T}l_{\gamma_S,S,T}(\theta)\right) l_{\gamma_S,S,T}(\theta).$$

Define $\Omega(\theta) := G'(\theta)V^{-1}(\theta)G(\theta)$. Then, NM-9.2 gives $\sqrt{T}\left(l_{\beta.\gamma_S,S,T}(\theta) - E\left[l_{\beta.\gamma_S,S,T}(\theta)\right]\right) \xrightarrow{d} N(0, \Xi(\theta))$

$$\begin{aligned} \text{where } \ \Xi_S(\theta) \ &:= \ \left(R_S^{1\prime}\Omega(\theta)R_S^1\right) - \left(R_S^{1\prime}\Omega(\theta)S_S^1\right)\left(S_S^{1\prime}\Omega(\theta)S_S^1\right)^{-1}\left(S_S^{1\prime}\Omega(\theta)R_S^1\right) \\ &= \ R_S^{1\prime}G'(\theta)V^{-1/2\prime}(\theta)\left(I_{d_g} - P\left(V^{-1/2}(\theta)G(\theta)S_S^1\right)\right)V^{-1/2}(\theta)G(\theta)R_S^1. \end{aligned}$$

Using the definitions of $G(\theta)$ and $V(\theta)$, the feasible version for $l_{\beta.\gamma_S,S,T}(\theta)$ (as is $\widehat{l}_T(\theta)$ for $l_T(\theta)$) is:

$$\widehat{l}_{\beta.\gamma_S,S,T}(\theta) = R_S^{1\prime}\widehat{G}_T'(\theta)\widehat{V}_T^{-1/2\prime}(\theta)\left(I_{d_g} - P\left(\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)S_S^1\right)\right)\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta),$$

and, similarly, $\widehat{\Xi}_T(\theta)$ for $\Xi_T(\theta)$. Then, the statistic in Chaudhuri and Zivot (2011) would be defined as:

$$\begin{aligned} LM_{T,S}^{\mathrm{ES}}(\theta) &:= T \times \widehat{l}'_{\beta.\gamma_S,T}(\theta)\ \widehat{\Xi}_{S,T}^{-1}(\theta)\ \widehat{l}_{\beta.\gamma_S,T}(\theta) \\ &= T \times \left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)' P\left(\left(I_{d_g} - P\left(\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)S_S^1\right)\right)\widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)R_S^1\right)\left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right). \end{aligned}$$

Chaudhuri and Zivot (2011) noted that Remark 5 from Section 3 is equally applicable to $LM_{T,S}^{\mathrm{ES}}(\theta)$.

Appendix B.3: Proofs of Lemmas 1 and 2 from Section 3:

We will repeatedly use the following relations that follow since $A_S = [R', S']'$ and $A_S^{-1} = [R_S^1, S_S^1]$:

$$RR_S^1 = I_{d_R},\ RS_S^1 = 0,\ SR_S^1 = 0,\ SS_S^1 = I_{d_\theta - d_R} \ \text{ and } \ R_S^1R + S_S^1S = I_{d_\theta}. \tag{29}$$

We will suppress the dependence of the quantities on θ to avoid notational clutter.

Proof of Lemma 1: Consider any $(d_\theta - d_R) \times d_\theta$ full row-rank matrix S in (3) such that $[R', S']'$ is nonsingular. Let ζ be a $d_\theta \times (d_\theta - d_R)$ matrix whose columns form a basis for the null space of R. Therefore, since $RS_S^1 = 0$ by (29) while S_S^1 is full column-rank by definition, we have $S_S^1 = \zeta B_S$ for some $(d_\theta - d_R) \times (d_\theta - d_R)$ nonsingular matrix B_S. Therefore, for any two such choices of $S = S^*, S^\dagger$, we have the corresponding $S_{S^*}^1 = \zeta B_{S^*}$ and $S_{S^\dagger}^1 = \zeta B_{S^\dagger}$ for some $(d_\theta - d_R) \times (d_\theta - d_R)$ nonsingular matrices B_{S^*} and $B_{S^\dagger}$. This implies that $S_{S^*}^1 = S_{S^\dagger}^1 B$ where $B = B_{S^\dagger}^{-1}B_{S^*}$ is $(d_\theta - d_R) \times (d_\theta - d_R)$ and nonsingular.

Now for any $d_\theta \times d_\theta$ nonsingular matrix $M = [M_1, M_2]$, where M_1 is $d_\theta \times d_R$, define:

$$\Phi_T(M) \;:=\; T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T M\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right), \tag{30}$$

$$\Phi_{1.2,T}(M) \;:=\; T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\left(I_{d_g} - P\left(\widehat{V}_T^{-1/2}\widehat{G}_T M_2\right)\right)\widehat{V}_T^{-1/2}\widehat{G}_T M_1\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right), \tag{31}$$

$$\Phi_{2,T}(M_2) \;:=\; T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T M_2\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right), \tag{32}$$

and note that, by construction:

(i) $\Phi_T(M) = \Phi_T(I_{d_\theta})$ since M is nonsingular,

(ii) $\Phi_T(M) = \Phi_{1.2,T}(M) + \Phi_{2,T}(M_2)$ since M is partitioned as $M = [M_1, M_2]$,

(iii) $\Phi_{2,T}(M_2) = \Phi_{2,T}(M_2 B)$ since B is a $(d_\theta - d_R) \times (d_\theta - d_R)$ nonsingular matrix.

In the above, now take $M = M^*, M^\dagger$ where $M^* = [R^1_{S^*}, S^1_{S^*}]$ and $M^\dagger = [R^1_{S^\dagger}, S^1_{S^\dagger}]$ correspond to the two choices $S = S^*, S^\dagger$ respectively. Thus we obtain:

$$\begin{aligned}
\Phi_T(M^*) &= \Phi_T(M^\dagger) \quad \text{[by (i)]}\\
\Phi_{1.2,T}(M^*) + \Phi_{2,T}(M_2^*) &= \Phi_{1.2,T}(M^\dagger) + \Phi_{2,T}(M_2^\dagger) \quad \text{[by (ii)]}\\
\Phi_{1.2,T}(M^*) &= \Phi_{1.2,T}(M^\dagger) \quad \text{[by (iii), since } M_2^* := S^1_{S^*} = S^1_{S^\dagger}B =: M_2^\dagger B\text{]}.
\end{aligned}$$

Thus, by its definition, we obtain that $LM^{\text{ES}}_{T,S^*}(\theta) = \Phi_{1.2,T}(M^*) = \Phi_{1.2,T}(M^\dagger) = LM^{\text{ES}}_{T,S^\dagger}\theta)$. ■

Proof of Lemma 2: Define $\widehat{\Omega}_T := \widehat{G}_T'(\theta)\widehat{V}_T^{-1}(\theta)\widehat{G}_T(\theta)$. Thus, the null space of $R\widehat{\Omega}_T^{-1}$ is of dimension $d_\theta - d_R$ since R is full row rank and $\widehat{\Omega}_T$ is nonsingular. Now, consider a $(d_\theta - d_R) \times d_\theta$ matrix S whose rows form the basis for the null space of $R\widehat{\Omega}_T^{-1}$. Note that, this specific choice of S and, hence, the related quantities R^1_S and S^1_S obtained from it all depend on the *given* T in the statement of the lemma.

Claim 1: With this S, we have a nonsingular $A_S := [R', S']'$ in (3).

Proof: Suppose not. Then, the full row-rank of $R = [R_1', \ldots, R_{d_R}']'$ implies that there exists a $(d_\theta - d_R) \times 1$ vector $c \neq 0$ such that $R_1 = \sum_{j=2}^{d_R} a_j R_j + c'S$ for some scalar coefficients $a_2, \ldots, a_{d_R}$. Since $\widehat{\Omega}_T^{-1}$ is positive definite, it means that for this $c \neq 0$, we have $R_1\widehat{\Omega}_T^{-1} = \sum_{j=2}^{d_R} a_j R_j \widehat{\Omega}_T^{-1} + c'S\widehat{\Omega}_T^{-1}$. Post-multiply both sides by S' and note that the rows of S belong in the null space of $R\widehat{\Omega}_T^{-1}$, i.e. $R_j\widehat{\Omega}_T^{-1}S' = 0$ for $j = 1, \ldots, d_R$. Hence, it follows that $0 = c'S\widehat{\Omega}_T^{-1}S'$. Since $S\widehat{\Omega}_T^{-1}S'$ is positive definite (as $\widehat{\Omega}_T^{-1}$ is positive definite and as the rows of S are linearly independent), this is only possible if $c = 0$, which contradicts our supposition. Therefore, Claim 1 is true. ■

Claim 2: $R\widehat{\Omega}_T^{-1}S' = 0$ if and only if $R^{1\prime}_S\widehat{\Omega}_T S^1_S = 0$.

Proof: We use (29) repeatedly in this proof. Post-multiply $R_S^{1\prime}\widehat{\Omega}_T S_S^1 = 0$ by S to get $R_S^{1\prime}\widehat{\Omega}_T S_S^1 S = 0$, i.e., $R_S^{1\prime}\widehat{\Omega}_T(I_{d_\theta} - R_S^1 R) = 0$ by (29). Hence,

$$R = (R_S^{1\prime}\widehat{\Omega}_T R_S^1)^{-1} R_S^{1\prime}\widehat{\Omega}_T. \tag{33}$$

Similarly obtain $S = (S_S^{1\prime}\widehat{\Omega}_T S_S^1)^{-1} S_S^{1\prime}\widehat{\Omega}_T$. Thus, $R\widehat{\Omega}_T^{-1}S' = (R_S^{1\prime}\widehat{\Omega}_T R_S^1)^{-1}(R_S^{1\prime}\widehat{\Omega}_T S_S^1)(S_S^{1\prime}\widehat{\Omega}_T S_S^1)^{-1}$. Hence, $R\widehat{\Omega}_T^{-1}S' = 0$ if and only if $R_S^{1\prime}\widehat{\Omega}_T S_S^1 = 0$, once again by using the positive definiteness of $\widehat{\Omega}_T$. ■

Thus, using this specific choice of S for which $R\widehat{\Omega}_T^{-1}S' = 0$ and hence $R_S^{1\prime}\widehat{\Omega}_T S_S^1 = 0$, we obtain from the definition of $LM_{T,S}^{\mathrm{ES}}(\theta)$ that $LM_{T,S}^{\mathrm{ES}}(\theta) = T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T R_S^1\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right)$. On the other hand, (4) gives:

$$\begin{aligned} LM_T(\theta) &= T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T\widehat{\Omega}_T^{-1}R'\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right) \\ &= T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T R_S^1 (R_S^{1\prime}\widehat{\Omega}_T R_S^1)^{-1}\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right) \end{aligned}$$

by using (33). However, since $(R_S^{1\prime}\widehat{\Omega}_T R_S^1)^{-1}$ is nonsingular, we have, by the construction of the projection matrix $P(.)$, that $P\left(\widehat{V}_T^{-1/2}\widehat{G}_T R_S^1(R_S^{1\prime}\widehat{\Omega}_T R_S^1)^{-1}\right) = P\left(\widehat{V}_T^{-1/2}\widehat{G}_T R_S^1\right)$. Therefore, $LM_T(\theta) = T \times \left(\widehat{V}_T^{-1/2}\bar{g}_T\right)' P\left(\widehat{V}_T^{-1/2}\widehat{G}_T R_S^1\right)\left(\widehat{V}_T^{-1/2}\bar{g}_T\right) = LM_{T,S}^{\mathrm{ES}}(\theta)$ (see above for the last equality). The desired result now follows from Lemma 1 for any general choice of S in (3) such that $[R', S']'$ is nonsingular. ■

Remark 20: The particular choice of S employed to facilitate the proof of Lemma 2 has an interesting interpretation. To see it, consider the analogous population version of S, i.e., S such that $R\Omega^{-1}S' = 0$. Similar to the proof of *Claim 1* above, it can be shown that $[R', S']'$ is nonsingular. Similar to the proof of *Claim 2* above, it can be shown that $R\Omega^{-1}S' = 0$ if and only if $R_S^{1\prime}\Omega S_S^1 = 0$, where the R_S^1 and S_S^1 correspond to this particular choice of S. Now, note from the discussion in Appendix B.2 that with this particular choice of S, the score for β, i.e., $l_{\beta,S,T}(\theta^0)$ is identical to the efficient score for β, i.e., $l_{\beta.\gamma_S,S,T}(\theta^0)$. In other words, this particular choice of S in the re-parameterization (3) directly makes the scores for β and γ_S uncorrelated and, by asymptotic normality, asymptotically independent. A followup along this line in the case of nonlinear null hypotheses is the topic of our ongoing research.

B.4 $\widetilde{LM}_T(\widetilde{\theta}_T) = LM_T(\widetilde{\theta}_T)$

From (30)-(32) and the definition in (3) it follows that $\widetilde{LM}_T(\theta) = LM_T(\theta) + \Phi_{2,T}(S_S^1, \theta)$ for all θ where the underlying quantities are defined. (Note that, by $\Phi_{2,T}(S_S^1, \theta)$ we mean $\Phi_{2,T}(S_S^1)$ with $\bar{g}_T$, $\widehat{G}_T$ and $\widehat{V}_T$ evaluated at θ.) Now, by the definition of the $\widetilde{\theta}_T$, i.e., $(R_S^1 r_0 + S_S^1\widetilde{\gamma}_T)$ where $\widetilde{\gamma}_T$ is the GMM estimator of γ under the restriction that $\beta = r_0$, it follows from the first order condition of the GMM optimization problem that $\Phi_{2,T}(S_S^1, \widetilde{\theta}_T) = 0$. This is because $\Phi_{2,T}(S_S^1, \theta)$ is simply a quadratic form of the first derivative of the GMM objective function with respect to γ_S, which is zero when evaluated at $\widetilde{\theta}_T$. Thus, $\widetilde{LM}_T(\widetilde{\theta}_T) = LM_T(\widetilde{\theta}_T)$. ■

Supplemental Appendix C: Proofs and clarifications for Section 4

C.1 Two useful lemmas

Since we use (have used) the following result in Lemma 10 often, let us state it here for reference.

Lemma 10 *Let X be an $a \times b$ matrix, and P and Q be $a \times a$ and $b \times b$ nonsingular matrices. Then, rank$(X) =$ rank$(PX) =$ rank(XQ).*

Proof: $\text{rank}(X) \geq \text{rank}(PX) \geq \text{rank}(P^{-1}PX) = \text{rank}(X) \geq \text{rank}(XQ) \geq \text{rank}(XQQ^{-1}) = \text{rank}(X)$. ■

Lemma 11 lists a set of intermediate results useful for proving the results in the main text.

Lemma 11 *Let assumptions O and N hold. Consider a sequence $\{\theta_T = R_S^1 r_0 + S_S^1 \gamma_{S,T} : T \geq 1\}$ where r_0 satisfies (14) and $\{\gamma_{S,T} : T \geq 1\}$ is such that θ_T satisfies (17). Then, the following results hold as $T \to \infty$:*

(a) $\widehat{V}_T(\theta_T) \xrightarrow{P} V(\theta^0) \equiv V$.

(b) $\widehat{V}_{Gg,T}(\theta_T) \xrightarrow{P} V_{Gg}(\theta^0) \equiv V_{Gg}$.

(c) $\bar{G}_T(\theta_T)\Pi_{\rho_\theta} D_{T,\rho_\theta} \xrightarrow{P} G^*$ *where* Π_{ρ_θ}, D_{T,ρ_θ} *and* G^* *are as defined in* (22), (23) *and* (25) *respectively.*

(d) $\sqrt{T}\bar{g}_T(\theta_T) = \sqrt{T}\bar{g}_T(\theta^0) + G^*\mu_{T,\theta} + o_p(1)$ *where* G^* *and* $\mu_{T,\theta}$ *are as defined in* (25) *and* (17) *respectively.*

(e) $\left[\widehat{V}_{1,g,T}(\theta_T)\widehat{V}_T^{-1}(\theta_T)\bar{g}_T(\theta_T), \ldots, \widehat{V}_{d_\theta,g,T}(\theta_T)\widehat{V}_T^{-1}(\theta_T)\bar{g}_T(\theta_T)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} = o_p(1)$ *(a $d_g \times d_\theta$ matrix).*

(f) $\widehat{G}_T(\theta_T)\Pi_{\rho_\theta} D_{T,\rho_\theta} \xrightarrow{P} G^*$ *where* Π_{ρ_θ}, D_{T,ρ_θ} *and* G^* *are as defined in* (22), (23) *and* (25) *respectively.*

Proof: (a) and (b) follow by assumption N8 since $\theta_T = \theta^0 + o_p(1)$.

(c) We prove it working term-by-term in the following decomposition:

$$\begin{aligned}
& \bar{G}_T(\theta_T)\Pi_{\rho_\theta} D_{T,\rho_\theta} \\
= \; & \left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} + \sqrt{T}\left[\bar{G}_T(\theta^0) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\right]\frac{\Pi_{\rho_\theta} D_{T,\rho_\theta}}{\sqrt{T}} + \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\Pi_{\rho_\theta} D_{T,\rho_\theta}. \quad (34)
\end{aligned}$$

From the definitions in (22) and (23) it follows that $\Pi_{\rho_\theta} D_{T,\rho_\theta} = o(\sqrt{T})$ by N3, and hence using N6 it follows that the second term on the right hand side (RHS) of (34) is $o_p(1)$. On the other hand, (24) and (25) imply that the third term on the RHS of (34) converges to G^* by construction.

To complete the proof, now we show that the first term on the RHS of (34) is $o_p(1)$. We deviate from Antoine and Renault (2012) in the treatment of this term, and the result thus obtained has implications in terms of the allowable weakness of identification [see the part of Remark 9 that led to footnote 8]. Let $\bar{G}_{T,i}(\theta) := \frac{\partial}{\partial\theta_i}\bar{g}_T(\theta)$ denote the i-th column of $\bar{G}_T(\theta)$ for $i = 1, \dots, d_\theta$ (recall that $\theta = (\theta_1, \dots, \theta_{d_\theta})'$). Therefore, with a bad but common abuse of notation in denoting the mean values element by element, we obtain by a mean value expansion of $\bar{G}_{T,i}(\theta_T)$ around $\bar{G}_{T,i}(\theta^0)$ for $i = 1, \dots, d_\theta$ that:

$$\begin{aligned}
&\left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} \\
= &\left[\left\{\frac{\partial}{\partial\theta'}\bar{G}_{T,1}(\theta_T(\theta_1))\right\}(\theta_T - \theta^0), \dots, \left\{\frac{\partial}{\partial\theta'}\bar{G}_{T,d_\theta}(\theta_T(\theta_{d_\theta}))\right\}(\theta_T - \theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} \\
= &\left[\left\{\frac{\partial}{\partial\theta_1}\bar{G}_T(\theta_T(\theta_1))\right\}(\theta_T - \theta^0), \dots, \left\{\frac{\partial}{\partial\theta_{d_\theta}}\bar{G}_T(\theta_T(\theta_{d_\theta}))\right\}(\theta_T - \theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} \qquad (35)
\end{aligned}$$

by twice interchanging the order in which the derivatives are taken in each of the d_θ columns. Note that, for $i = 1, \dots, d_\theta$, we used $\theta_T(\theta_i)$ (such that $\|\theta_T(\theta_i) - \theta^0\| \le \|\theta_T - \theta^0\|$) to denote the mean value in the first equality of the above equation. Recalling that $\mu_{T,\theta} = \sqrt{T}D_{T,\rho_\theta}^{-1}\Pi'_{\rho_\theta}(\theta_T - \theta^0)$ by (17), define $U_{T,i}$ for $i = 1, \dots, d_\theta$ as the $d_g \times d_\theta$ matrix with

$$\left\{\frac{\partial}{\partial\theta_i}\bar{G}_T(\theta_T(\theta_i))\right\}(\theta_T - \theta^0) = \left\{\frac{\partial}{\partial\theta_i}\bar{G}_T(\theta_T(\theta_i))\frac{\sqrt{T}}{\lambda_{T,l}}\right\}\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\mu_{T,\theta}\frac{\lambda_{T,l}}{\sqrt{T}\lambda_{T,j_1}}$$

in the i-th column and zero everywhere else. [See Remark 9 for λ_{T,j_1}.] Therefore, (35) implies that:

$$\left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta} = \sum_{i=1}^{d_\theta} U_{T,i}\Pi_{\rho_\theta}D_{T,\rho_\theta},$$

and thus,

$$\begin{aligned}
&\left\|\left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta}\right\| \\
\le &\sum_{i=1}^{d_\theta}\|U_{T,i}\| \times \|\Pi_{\rho_\theta}D_{T,\rho_\theta}\| \\
\le &\sum_{i=1}^{d_\theta}\left\|\frac{\partial}{\partial\theta_i}\bar{G}_T(\theta_T(\theta_i))\frac{\sqrt{T}}{\lambda_{T,l}}\right\| \times \left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\| \times \|\mu_{T,\theta}\| \times \left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\| \frac{\sqrt{T}\lambda_{T,l}}{\lambda_{T,j_1}^2\sqrt{T}} \\
\le &\sum_{i=1}^{d_\theta}\sup_\theta\left\{\left\|\frac{\sqrt{T}}{\lambda_{T,l}}\frac{\partial}{\partial\theta_i}\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right\| + \left\|\frac{\sqrt{T}}{\lambda_{T,l}}\frac{\partial}{\partial\theta_i}\left[\bar{G}_T(\theta) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right]\right\|\right\} \times \|\mu_{T,\theta}\| \times \left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|^2 \frac{\lambda_{T,l}}{\lambda_{T,j_1}^2} \\
= &\ o_p(1)
\end{aligned}$$

since, on the third line from above, the order of magnitude of the terms (from left to right) inside the sum is respectively: (i) $\sup_\theta\left\|\frac{\sqrt{T}}{\lambda_{T,l}}\frac{\partial}{\partial\theta_i}\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right\| = O(1)$ by N3 and N4; (ii) $\sup_\theta\left\|\frac{\sqrt{T}}{\lambda_{T,l}}\frac{\partial}{\partial\theta_i}\left[\bar{G}_T(\theta) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right]\right\| = o_p(1)$ by N3 and N7(a), (iii) $\|\mu_{T,\theta}\| = O_p(1)$ by (17); (iv) $\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\| = O(1)$ by N3, (22) and (23);

and (v) $\frac{\lambda_{T,l}}{\lambda^2_{T,j_1}} = o(1)$ by N7(b) [also see Remark 9].

(d) A mean value expansion (with similar abuse of notation as above to denote the mean value $\bar{\theta}_T$) gives $\sqrt{T}\bar{g}_T(\theta_T) = \sqrt{T}\bar{g}_T(\theta^0) + \bar{G}_T(\bar{\theta}_T)\sqrt{T}(\theta_T - \theta^0) = \sqrt{T}\bar{g}_T(\theta^0) + \bar{G}_T(\bar{\theta}_T)\Pi_{\rho_\theta} D_{T,\rho_\theta}\mu_{T,\theta} = \sqrt{T}\bar{g}_T(\theta^0) + G^*\mu_{T,\theta} + o_p(1)$ where the second equality uses (17) and the last one uses the result from Lemma 11(c).

(e) The result follows by Lemma 11 (a), (b), (d) since $\Pi_{\rho_\theta} D_{T,\rho_\theta} = o(\sqrt{T})$ by N3 and $\sqrt{T}\bar{g}_T(\theta^0) = O_p(1)$.

(f) The result follows by Lemma 11 (c) and (e). ■

C.2 Clarification and details regarding footnote 8:

We briefly illustrate the said tradeoff by showing that if one strengthens the smoothness assumption by extending it to the second derivative, then this allows to weaken the rate assumption in N7(b). Since, at this point the definition of λ_{T,j_1} is already stated, and since Lemma 11 works with λ_{T,j_1} instead of $\lambda_{T,1}$, the discussion below uses λ_{T,j_1}. All we do hold if assumptions are maintained in terms of $\lambda_{T,1}$.

From Lemma 11, it is clear that the discussion here is pertinent mainly to part Lemma 11 (c) [see the proof of Lemma 11 (d)-(f)]. Indeed, the only part of (c) that needs attention is where we show that $\left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta} D_{T,\rho_\theta}$, i.e., the first term on the RHS of (34), is $o_p(1)$. So, let us focus on this.

Remark 21: If $g(Z_t;\theta)$ is linear in θ, as in linear instrumental variables models, then this is trivially true since $\left[\bar{G}_T(\theta) - \bar{G}_T(\theta^0)\right] \equiv 0$ for all θ. So, let us focus on a $g(Z_t;\theta)$ that is nonlinear in θ.

Now, to accommodate for more smoothness we extend assumption N6 as N6' to the second derivative, and replace N7 by N7' as follows. (Assumptions N1-N5 and N8 remain the same.)

Assumption N6': (a one-time assumption for this clarification only)

(a) $\frac{\partial}{\partial\theta'}\psi_T(\theta^0) = \sqrt{T}\left[\bar{G}_T(\theta^0) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\right] = O_p(1)$. (This was the original N6.)

(b) For $i = 1,\ldots,d_\theta$: $\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta'}\psi_T(\theta^0) = \sqrt{T}\frac{\partial}{\partial\theta_i}\left[\bar{G}_T(\theta^0) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\right] = O_p(1)$. (This is the extension.)

Assumption N7': (a one-time assumption for this clarification only)

(a) $\rho(\theta)$ is thrice continuously differentiable in $\theta \in \mathcal{N}(\theta^0)$. $g(z;\theta)$ is thrice differentiable in $\theta \in \mathcal{N}(\theta^0)$ for each $z \in \mathbb{R}^{d_z}$ and $\sup_{\theta\in\mathcal{N}(\theta^0)}\left\|\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\left[\bar{G}_T(\theta) - \frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right]\right\| = o_p(\lambda_{T,l}/\sqrt{T})$ for $i,k = 1,\ldots,d_\theta$.

(b) λ_{T,j_1} from (23) satisfies $\lambda^3_{T,j_1}/\lambda_{T,l} \to \infty$ as $T \to \infty$.

Comparing assumption N7 with N7' reveals the tradeoff in terms of parts (a) and (b) of these assumptions. We note that similar tradeoffs can be generated by working with higher order derivatives.

For clarity, specify further structure but without loss of generality. First, for $i = 1,\ldots,d_\theta$, define $\Pi_{\rho_\theta,i}$ and $D_{T,\rho_\theta,i}$ by the UBT-Construction like that in (22) and (23), but this time, by taking

$$W_T = \left[\frac{\partial}{\partial\theta_i}\rho'_{\theta,1}(\theta^0),\ldots,\frac{\partial}{\partial\theta_i}\rho'_{\theta,l}(\theta^0)\right] = \left(I^*\frac{\partial}{\partial\theta_i}\rho_\theta(\theta^0)\right)'$$

(instead of $W_T = \left[\rho'_{\theta,1}(\theta^0), \ldots, \rho'_{\theta,l}(\theta^0)\right] = \left(I^* \rho_\theta(\theta^0)\right)'$) not depending on T in the UBT-Construction. The corresponding quantities with full column-rank, and thus also the elements of $D_{T,\rho_\theta,i}$ will change. Indeed, no full-rank conditions are required, and instead, for the purpose of this proof, the only properties we will require are: For $i = 1, \ldots d_\theta$,

$$
\begin{aligned}
\left(I^{*'} \left\{ \frac{\partial}{\partial \theta_i} I^* \frac{\Lambda_T}{\sqrt{T}} I^{*'} I^* \rho_\theta(\theta^0) \right\} \Pi_{\rho_\theta,i} D_{T,\rho_\theta,i} \right) &= O(1), &(36)\\
\Pi_{\rho_\theta,i} D_{T,\rho_\theta,i} &= o(\sqrt{T}) &(37)
\end{aligned}
$$

and these will not change since (36) holds by the construction of $D_{T,\rho_\theta,i}$, while (37) follows from N3.

Start from (35). All we do below is to tease out further structure in the non-zero (i.e.,the i-th) column of $U_{T,i}$ (defined below (35)) so that assumption N7' could be effectively used to show that $\left[\bar{G}_T(\theta_T) - \bar{G}_T(\theta^0)\right] \Pi_{\rho_\theta} D_{T,\rho_\theta}$, i.e., the first term on the RHS of (34) is $o_p(1)$. With this purpose in mind, for each $i = 1, \ldots, d_\theta$, consider a further mean value expansion (with similar abuse of notation, and this time using $\theta_T(\theta_i^k)$ to denote the mean value such that $\|\theta_T(\theta_i^k) - \theta^0\| \leq \|\theta_T(\theta_i) - \theta^0\| \leq \|\theta_T - \theta^0\|$ for $k = 1, \ldots, d_\theta$):

$$
\begin{aligned}
& \left\{ \frac{\partial}{\partial \theta_i} \bar{G}_T(\theta_T(\theta_i)) \right\} (\theta_T - \theta^0) \\
= & \left\{ \frac{\partial}{\partial \theta_i} \bar{G}_T(\theta^0) \right\} (\theta_T - \theta^0) \\
& + \left[\left\{ \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_1} \bar{G}_T(\theta_T(\theta_i^1)) \right\} (\theta_T(\theta_i) - \theta^0), \ldots, \left\{ \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial \theta_{d_\theta}} \bar{G}_T(\theta_T(\theta_i^{d_\theta})) \right\} (\theta_T(\theta_i) - \theta^0) \right] (\theta_T - \theta^0)
\end{aligned}
$$

by similar (to above) interchange in the order of the derivatives. Since $\mu_{T,\theta} = \sqrt{T} D_{T,\rho_\theta}^{-1} \Pi'_{\rho_\theta} (\theta_T - \theta^0)$ by (17), it follows that:

$$
\begin{aligned}
\left\{ \frac{\partial}{\partial \theta_i} \bar{G}_T(\theta^0) \right\} (\theta_T - \theta^0) = & \underbrace{\left(I^{*'} \left\{ \frac{\partial}{\partial \theta_i} I^* \frac{\Lambda_T}{\sqrt{T}} I^{*'} I^* \rho_\theta(\theta^0) \right\} \Pi_{\rho_\theta,i} D_{T,\rho_\theta,i} \right) \mu_{T,\theta} \frac{1}{\sqrt{T}}}_{= u_{a,T,i} \text{ (say)}} \\
& + \underbrace{\left(\frac{\partial}{\partial \theta_i} \sqrt{T} \left(\bar{G}_T(\theta^0) - \frac{\Lambda_T}{\sqrt{T}} \rho_\theta(\theta^0) \right) \right) \left(\frac{\Pi_{\rho_\theta,i} D_{T,\rho_\theta,i}}{\sqrt{T}} \right) \mu_{T,\theta} \frac{1}{\sqrt{T}}}_{= u_{b,T,i} \text{ (say)}}
\end{aligned}
$$

for $i = 1, \ldots, d_\theta$. Define the $d_g \times d_\theta$ matrices $U_{a,T,i}$ and $U_{b,T,i}$ such that all their columns are zeros, except for the i-th column, which for them is $u_{a,T,i}$ and $u_{b,T,i}$ respectively. Do this for all $i = 1, \ldots, d_\theta$.

On the other hand, for the notation-abused quantity $\theta_T(\theta_i)$, define $\mu_{T,\theta(i)} := \sqrt{T} D_{T,\rho_\theta}^{-1} \Pi'_{\rho_\theta} (\theta_T(\theta_i) - \theta^0)$ where $\|\mu_{T,\theta(i)}\| \leq \|\mu_{T,\theta}\|$ by construction for $i = 1, \ldots, d_\theta$ (recall that $\mu_{T,\theta} = \sqrt{T} D_{T,\rho_\theta}^{-1} \Pi'_{\rho_\theta} (\theta_T - \theta^0)$ by (17)). Now for each $i = 1, \ldots, d_\theta$, define the $d_g \times d_\theta$ matrices $U_{c,T,i,k}$ for $k = 1, \ldots, d_\theta$ such that all the columns of $U_{c,T,i,k}$ are zeros, except for the k-th column which is:

$$\left\{\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\bar{G}_T(\theta_T(\theta_i^k))\right\}(\theta_T(\theta_i)-\theta^0)=\left\{\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\bar{G}_T(\theta_T(\theta_i^k))\frac{\sqrt{T}}{\lambda_{T,l}}\right\}\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\mu_{T,\theta(i)}\frac{\lambda_{T,l}}{\sqrt{T}\lambda_{T,j_1}}.$$

Therefore, it follows that $U_{T,i}$ (defined below (35)) can be written as:

$$U_{T,i}=U_{a,T,i}+U_{b,T,i}+\left(\sum_{k=1}^{d_\theta}U_{c,T,i,k}\right)(\theta_T-\theta^0)=U_{a,T,i}+U_{b,T,i}+\left(\sum_{k=1}^{d_\theta}U_{c,T,i,k}\right)\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\mu_{T,\theta}\frac{1}{\lambda_{T,j_1}}.$$

And, therefore,

$$\begin{aligned}
&\left\|\left[\bar{G}_T(\theta_T)-\bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta}D_{T,\rho_\theta}\right\|\\
\leq\;&\sum_{i=1}^{d_\theta}\|U_{T,i}\|\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|\\
\leq\;&\sum_{i=1}^{d_\theta}\|U_{a,T,i}\|\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|+\sum_{i=1}^{d_\theta}\|U_{b,T,i}\|\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|\\
&+\sum_{i=1}^{d_\theta}\sum_{k=1}^{d_\theta}\|U_{c,T,i,k}\|\times\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|\times\frac{\|\mu_{T,\theta}\|}{\lambda_{T,j_1}}\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|.
\end{aligned}$$

Since $\|u_{a,T,i}\|=O(1/\sqrt{T})$ by its definition and using (36), it follows that $\sum_{i=1}^{d_\theta}\|U_{a,T,i}\|\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|=o_p(1)$ by using (37). Since $\|u_{b,T,i}\|=O(1/\sqrt{T})$ by its definition and using N6' and (37), it follows that $\sum_{i=1}^{d_\theta}\|U_{b,T,i}\|\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|=o_p(1)$ by using (37). Finally, note that $\sum_{i=1}^{d_\theta}\sum_{k=1}^{d_\theta}\|U_{c,T,i,k}\|\times\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|\times\frac{\|\mu_{T,\theta}\|}{\lambda_{T,j_1}}\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|=o_p(1)$ since, collecting similar terms together,

$$\begin{aligned}
&\|U_{c,T,i,k}\|\times\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|\times\frac{\|\mu_{T,\theta}\|}{\lambda_{T,j_1}}\times\|\Pi_{\rho_\theta}D_{T,\rho_\theta}\|\\
\leq\;&\sup_\theta\left\{\left\|\frac{\Lambda_T}{\sqrt{T}}\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\rho_\theta(\theta)\times\frac{\sqrt{T}}{\lambda_{T,l}}\right\|+\left\|\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\left[\bar{G}_T(\theta)-\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right]\frac{\sqrt{T}}{\lambda_{T,l}}\right\|\right\}\\
&\times\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|^3\times\|\mu_{T,\theta(i)}\|\times\|\mu_{T,\theta}\|\times\frac{\lambda_{T,l}}{\lambda_{T,j_1}^3}\\
=\;&O_p(1)\times O(1)\times O_p(1)\times O_p(1)\times o(1)
\end{aligned}$$

term by term: (i) $\sup_\theta\left\|\frac{\Lambda_T}{\sqrt{T}}\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\rho_\theta(\theta)\times\frac{\sqrt{T}}{\lambda_{T,l}}\right\|+\left\|\frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_k}\left[\bar{G}_T(\theta)-\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta)\right]\frac{\sqrt{T}}{\lambda_{T,l}}\right\|=O(1)+o_p(1)$ by using N3 and N7'(a); (ii) $\left\|\frac{\Pi_{\rho_\theta}D_{T,\rho_\theta}\lambda_{T,j_1}}{\sqrt{T}}\right\|^3=O(1)$ by using N3, (22) and (23); (iii) $\|\mu_{T,\theta(i)}\|=O_p(1)$ by using (17) and the definition of $\mu_{T,\theta(i)}$; (iv) $\|\mu_{T,\theta}\|=O_p(1)$ by using (17); and (v) $\frac{\lambda_{T,l}}{\lambda_{T,j_1}^3}=o(1)$ by using N7'(b). Thus $\left[\bar{G}_T(\theta_T)-\bar{G}_T(\theta^0)\right]\Pi_{\rho_\theta}D_{T,\rho_\theta}=o_p(1)$.

This completes the announced demonstration in the clarification for footnote 8. ■

C.3 Proof of the results from Section 4

Proof of Lemma 3: The proof is based on the original work of Antoine and Renault (2012), Andrews and Guggenberger (2014), Andrews and Cheng (2014) and Cheng (2015), with suitable adjustments that are required by our setup. Let $\widehat{G}_T := \widehat{G}_T(\theta^0)$, $\widehat{V}_T := \widehat{V}_T(\theta^0)$. By M1 and M2, $\widehat{V}_T$ is positive definite with probability approaching one as $T \to \infty$. Thus, if defined, let $\widehat{V}_T^{-1/2}$ be such that $\widehat{V}_T^{-1/2'}\widehat{V}_T^{-1/2} = \widehat{V}_T^{-1}$ and let $\widehat{g}_T := \widehat{V}_T^{-1/2}\bar{g}_T(\theta^0)$ and $H_T := \widehat{V}_T^{-1/2}\widehat{G}_T$. Then, for T sufficiently large, (4) gives:

$$
\begin{aligned}
LM_T(\theta^0) &= T\widehat{g}_T' P\left(H_T\{H_T'H_T\}^{-1}R'\right)\widehat{g}_T \\
&= T\widehat{g}_T' P\left(H_T B_T \Upsilon_T \left\{(H_T B_T \Upsilon_T)'(H_T B_T \Upsilon_T)\right\}^{-1}\Upsilon_T B_T' R' \Pi_T^* D_T^*\right)\widehat{g}_T
\end{aligned}
$$

where $\Upsilon_T := \text{diag}(1/\delta_{T,1},\ldots,1/\delta_{T,p},\sqrt{T}1_{d_\theta-p})$, a $d_\theta \times d_\theta$ diagonal matrix, nonsingular for any given T. (1_c is the $1\times c$ vector $(1,\ldots,1)$.) Υ_T is $\text{diag}(1/\delta_{T,1},\ldots,1/\delta_{T,p})$ if $d_\theta = p$ and is $\text{diag}(\sqrt{T}1_{d_\theta-p})$ if $p=0$. For a given T, Π_T^* and D_T^* are $d_R\times d_R$ nonsingular matrices defined as follows.

Step 1: Definition of Π_T^* and D_T^*, and the asymptotic behavior of $\Upsilon_T B_T' R'\Pi_T^* D_T^*$

Under assumption M3(a) we can, without loss of generality, partition the set of elements $\delta_{T,1},\ldots,\delta_{T,p}$ into $m-1$ groups containing $p_1,p_2,\ldots,p_{m-1}$ elements respectively as $(\delta_{T,1},\ldots,\delta_{T,p_1})$, $(\delta_{T,\bar{p}_1+1},\ldots,\delta_{T,\bar{p}_2})$,..., $(\delta_{T,\bar{p}_{m-2}+1},\ldots,\delta_{T,\bar{p}_{m-1}})$ where $p_j\geq 0$ and $\bar{p}_j := \sum_{k=1}^{j} p_k$ for $j=1,\ldots,m-1$ and $m\in\{1,\ldots,p+1\}$ (let $p_m := d_\theta - p$; and when $p=0$ let $m=1$; and also, by construction, $\bar{p}_{m-1}=p$ and $\bar{p}_m = d_\theta$), such that:

$$
\delta_{T,\bar{p}_j} \neq o(\delta_{T,\bar{p}_j-p_j+1}) \text{ for } j=1,\ldots,m-1, \text{ and } \delta_{T,\bar{p}_j+1} = o(\delta_{T,\bar{p}_j}) \text{ for } j=1,\ldots,m-2. \tag{38}
$$

Taking $W_T := RB_T = [W_{T,1},\ldots,W_{T,m}]$ where $W_{T,j} := RB_{T,(\bar{p}_j-p_j+1:\bar{p}_j)}$ for $j=1,\ldots,m$, define $\Pi_T^* = [\Pi_{T,1}^*,\ldots,\Pi_{T,m}^*]$ as the Π_T matrix from the UBT-Construction in Appendix A.1.1. B_T is orthogonal for each T and also $B_T \to B$, which is nonsingular by M3(c). Therefore, by Lemma 10, quantities such as q_T and $c_{T,j_i,T}^*$ in the UBT-Construction have well defined limits as $T\to\infty$. Denote these limits as q and $c_{j_i}^*$ respectively, i.e., by dropping the subscript T, and note that $\sum_{i=1}^{q} c_{j_i}^* = d_R$.

Define $D_T^* = \text{diag}(\delta_{T,\bar{p}_{j_1}}1_{c_{j_1}^*},\ldots,\delta_{T,\bar{p}_{j_q}}1_{c_{j_q}^*})$ where we use the notation $\delta_{T,\bar{p}_{m-1}+1} = \ldots = \delta_{T,\bar{p}_m} = T^{-1/2}$ to allow for the possibility that $j_q = m$. D_T^* is a $d_R\times d_R$ nonsingular diagonal matrix for each T.

Therefore, as $T\to\infty$, it follows by M3(a) and (38), and then again using Lemma 10, that

$$
W^{*'} = \lim_{T\to\infty} \Upsilon_T B_T' R' \Pi_T^* D_T^* \tag{39}
$$

is a finite, non-random, $d_\theta \times d_R$ matrix with full column-rank d_R.[15]

[15]To see its full column-rank, use arguments similar to those below (23) along with M3(a) to obtain that for $W^{*'}$, its

The rest of the proof is completely based on Andrews and Guggenberger (2014).

Step 2: Asymptotic behavior of $H_T B_T \Upsilon_T$

Under (9), $\|\Delta_T\| \leq c \times \bar{c}$ for some $c > 0$ by M2. Then, it follows that:

$$\begin{aligned} V_T^{-1/2}\widehat{G}_T B_T \Upsilon_T &= V_T^{-1/2}\widehat{G}_T \left[B_{T,(1:p)}\Delta_{T,(1:p)}^{-1}, \sqrt{T}B_{T,(p+1:d_\theta)}\right] \\ &= V_T^{-1/2}G_T \left[B_{T,(1:p)}\Delta_{T,(1:p)}^{-1}, \sqrt{T}B_{T,(p+1:d_\theta)}\right] \\ &\quad + V_T^{-1/2}\sqrt{T}\left(\widehat{G}_T - G_T\right)\left[B_{T,(1:p)}(\sqrt{T}\Delta_{T,(1:p)})^{-1}, B_{T,(p+1:d_\theta)}\right]. \end{aligned}$$

By the orthogonality of B_T, it follows from the relation $V_T^{-1/2}G_T = C_{T,(1:d_R)}\Delta_T B_T'$ (obtained from (9)) and M3, that the first term on the right hand side of the above equation converges to $[C_{(1:p)}, C_{(p+1:d_\theta)}L]$. On the other hand, M1 and M2 give $\sqrt{T}\left(\widehat{G}_T - G_T\right) \xrightarrow{d} devec_{d_g}(\psi_G - V_{Gg}V^{-1}\psi) = O_p(1)$ which, crucially, is independent of ψ. Also M3 implies that $[B_{T,(1:p)}(\sqrt{T}\Delta_{T,(1:p)})^{-1}, B_{T,(p+1:d_\theta)}] \to [0, B_{(p+1:d_\theta)}]$ as $T \to \infty$. Thus, by M1, for the second term on the right hand side of the above equation, we now have that $V_T^{-1/2}\sqrt{T}\left(\widehat{G}_T - G_T\right)\left[B_{T,(1:p)}(\sqrt{T}\Delta_{T,(1:p)})^{-1}, B_{T,(p+1:d_\theta)}\right] \xrightarrow{d} [0, V^{-1/2}devec_{d_g}(\psi_G - V_{Gg}V^{-1}\psi)B_{(p+1:d_\theta)}]$. Since M2 implies that $\widehat{V}_T^{-1/2}V_T^{1/2} \xrightarrow{p} I_{d_g}$, it now follows, by combining the two terms, that

$$H_T B_T \Upsilon_T = \widehat{V}_T^{-1/2}\widehat{G}_T B_T \Upsilon_T = \left(\widehat{V}_T^{-1/2}V_T^{1/2}\right)V_T^{-1/2}\widehat{G}_T B_T \Upsilon_T \xrightarrow{d} G^* \tag{40}$$

where $G^* := [C_{(1:p)}, C_{(p+1:d_\theta)}L + V^{-1/2}devec_{d_g}(\psi_G - V_{Gg}V^{-1}\psi)B_{(p+1:d_\theta)}]$, as defined in M3(d).

Step 3: Asymptotic behavior of $LM_T(\theta^0)$

Therefore, $P(H_T B_T \Upsilon_T\{(H_T B_T \Upsilon_T)'(H_T B_T \Upsilon_T)\}^{-1}\Upsilon_T B_T' R' \Pi_T^* D_T^*) \xrightarrow{d} P(G^*(G^{*\prime}G^*)^{-1}W^{*\prime})$, a finite matrix with full column-rank d_R almost surely by (39), (40) and Lemma 10. Now, since M1 and M2 imply that $\sqrt{T}\widehat{g}_T \xrightarrow{d} V^{-1/2}\psi \sim N(0, I_{d_g})$, and since we have already noted the independence between ψ and G^*, it follows that $LM_T(\theta^0) \xrightarrow{d} \chi^2_{d_R}$. ■

Proof of Proposition 4: Let $\{\phi_{\gamma_S,T} : T \geq 1\}$ denote the sequence of indicator variables where $\phi_{\gamma_S,T} = 0$ if $CI_T(\gamma_S; \epsilon)$ contains γ_S^0, and $\phi_{\gamma_S,T} = 1$ otherwise. It is given that $CI_T(\gamma_S; \epsilon)$ has asymptotic coverage

columns from $(d_R - \sum_{i'=i}^{q} c_{j_{i'}}^*)$ to $(d_R - \sum_{i'=i}^{q} c_{j_{i'}}^* + c_{j_i}^*)$ for $i = 1, \ldots, q$ are represented by the $d_g \times c_{j_i}^*$ matrix:

$$\begin{aligned} &\left[(\delta_{\bar{p}_1}\mathrm{diag}(\delta_1^{-1}, \ldots, \delta_{\bar{p}_1}^{-1})B_{(1:p_1)}' R' \bar{\Pi}_1)', 0'\right]' \text{ if } j_i = 1, \\ &\left[0', (\delta_{\bar{p}_{j_i}}\mathrm{diag}(\delta_{\bar{p}_{j_i}-p_{j_i}+1}^{-1}, \ldots, \delta_{\bar{p}_{j_i}}^{-1})B_{(\bar{p}_{j_i}-p_{j_i}+1:\bar{p}_{j_i})}' R' \bar{\Pi}_{j_i})', 0'\right]' \text{ otherwise} \end{aligned}$$

(where the 0 denotes sub-matrices of zeros with number of rows, which can be zero, such that the number of rows of the corresponding big matrix is d_θ). Thus the non-zero blocks in such sets of columns (one block per set of columns) are: (i) at mutually non-overlapping positions (sets of rows); (ii) are finite by M1, M3(a); (iii) of full column-rank by Lemma 10, which tells that pre-multiplication by the nonsingular matrix $\delta_{\bar{p}_{j_i}}\mathrm{diag}(\delta_{\bar{p}_{j_i}-p_{j_i}+1}^{-1}, \ldots, \delta_{\bar{p}_{j_i}}^{-1})$ does not change the rank of $B_{(\bar{p}_{j_i}-p_{j_i}+1:\bar{p}_{j_i})}' R' \bar{\Pi}_{j_i}$. The latter has full column-rank $c_{j_i}^*$ for $i = 1, \ldots, q$ by (i) in the UBT-Construction. Therefore, full column-rank d_R of $W^{*\prime}$ follows by noting that $\sum_{i=1}^{q} c_{j_1}^* = d_R$. Note that the additional structure in M3(a), that Andrews and Guggenberger (2014) did not require, was imposed here precisely for this step [see sentence 4 in Remark 6].

$(1-\epsilon)$ when H_0 is true. Hence, $\lim_{T\to\infty} Pr_T(\phi_{\gamma_S,T}=0) \geq (1-\epsilon)$ where $Pr_T(.)$ denotes the probability of an event under F_T constrained by assumptions O and M1-M3 and when $\beta^0 = r_0$, equivalently, when $R\theta^0 = r_0$ [see Remark 22 and footnote 1 in that order]. Therefore, by construction:

$$\lim_{T\to\infty} Pr_T\left(\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) \leq LM_T(\theta^0)\right) \geq \lim_{T\to\infty} Pr_T(\phi_{\gamma_S,T}=0) \geq 1-\epsilon, \tag{41}$$

since for any $T \geq 1$, the event $\{\phi_{\gamma_S,T}=0\} \subseteq$ the event $\{\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) \leq LM_T(\theta^0)\}$.

Recall that the definition in (6) allows for the convention that $\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) = \infty$ if $CI_T(\gamma_S;\epsilon)$ is empty. Now, let $\{\phi_{\beta,T} : T \geq 1\}$ denote the sequence of indicator variables where $\phi_{\beta,T}=1$ if $\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) > \chi^2_{d_R}(1-\alpha)$, and $\phi_{\beta,T}=0$ otherwise. Therefore,

$$\begin{aligned}
& Pr_T\left(\phi_{\beta,T}=0\right) \\
= \;& Pr_T\left(\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) \leq \chi^2_{d_R}(1-\alpha)\right) \\
\geq \;& Pr_T\left(\left\{\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) \leq LM_T(\theta^0)\right\} \bigcap \left\{LM_T(\theta^0) \leq \chi^2_{d_R}(1-\alpha)\right\}\right) \\
= \;& 1 - Pr_T\left(\left\{\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) > LM_T(\theta^0)\right\} \bigcup \left\{LM_T(\theta^0) > \chi^2_{d_R}(1-\alpha)\right\}\right) \\
\geq \;& 1 - \left(Pr_T\left(\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) > LM_T(\theta^0)\right) + Pr_T\left(LM_T(\theta^0) > \chi^2_{d_R}(1-\alpha)\right)\right),
\end{aligned}$$

where the second line follows by the definition of $\phi_{\beta,T}$, the third line by the construction of the two-step projection test in (6), the fourth line by De Morgan's law, and the fifth line by Bonferroni's inequality. Taking limits on both sides gives:

$$\begin{aligned}
\lim_{T\to\infty} Pr_T\left(\phi_{\beta,T}=0\right) \;\geq\; & 1 - \lim_{T\to\infty} Pr_T\left(\inf_{\gamma_0\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0',\gamma_0')'\right) > LM_T(\theta^0)\right) \\
& - \lim_{T\to\infty} Pr_T\left(LM_T(\theta^0) > \chi^2_{d_R}(1-\alpha)\right) \\
\geq\; & 1 - (\epsilon+\alpha),
\end{aligned}$$

where the last line follows by (41) and Lemma 3. ■

Remark 22: Since the way it is stated in the statement of the proposition, the coverage probability of $CI_T(\gamma_S;\epsilon)$ is $(1-\epsilon)$, possibly under a larger class of distributions than F_T constrained by the assumptions O and M1-M3. This is the reason behind the inequality $\lim_{T\to\infty} Pr_T(\phi_{\gamma_S,T}=0) \geq (1-\epsilon)$. However, the confidence sets $CI_T(\gamma_S;\epsilon)$, e.g., $CI_T^{SW}(\gamma_S;r_0,\epsilon)$ defined in (10), that we actually mention [see Remark 2] are asymptotically similar and hence, for them, the above inequality will hold as an equality. ■

Proof of Lemma 5: (a) Utilizing the nonsingular matrices Π_{ρ_θ}, D_{T,ρ_θ}, Π_R and $D_{T,R}$ in (22),(23), (26) and (27) respectively, recall from (4) that $LM_T(\theta)$ can be written as:

$$LM_T(\theta) = T \times \left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)' P\left(\widehat{H}_T(\theta)(\widehat{H}_T'(\theta)\widehat{H}_T(\theta))^{-1}D_{T,\rho_\theta}\Pi_{\rho_\theta}' R'\Pi_R D_{T,R}\right)\left(\widehat{V}_T^{-1/2}(\theta)\bar{g}_T(\theta)\right)$$

where $\widehat{H}_T(\theta) := \widehat{V}_T^{-1/2}(\theta)\widehat{G}_T(\theta)\Pi_{\rho_\theta}D_{T,\rho_\theta}$. Now note that for θ_T defined in (17), we have:

(i) $\widehat{V}_T^{-1/2}(\theta_T) \xrightarrow{P} V^{-1/2}$ by N8 [see Lemma 11(a)] and $V_T^{-1/2}(\theta_T) \to V^{-1/2}$ by definition [also see N8];

(ii) $\widehat{V}_T^{-1/2}(\theta_T)\sqrt{T}\bar{g}_T(\theta_T) = V^{-1/2}[\sqrt{T}\bar{g}_T(\theta^0) + G^*\mu_{T,\theta}] + o_p(1)$ by (i) and Lemma 11(c); and the RHS is $O_p(1)$ by N2, N8, and the definition of $\mu_{T,\theta}$ in (17);

(iii) $\widehat{H}_T(\theta_T) \xrightarrow{P} V^{-1/2}G^*$ by (i) and Lemma 11(f).

(iv) $D_{T,\rho_\theta}\Pi_{\rho_\theta}' R'\Pi_R D_{T,R} \to R^{*'}$ by (28).

Note that the limiting quantities in (i), (iii) and (iv) are also finite and full column-rank by construction. Therefore, from (i)-(iv) it follows that

$$\begin{aligned} & LM_T(\theta_T) \\ = \; & \left(V^{-1/2}[\sqrt{T}\bar{g}_T(\theta^0) + G^*\mu_{T,\theta}]\right)' P\left(V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)\left(V^{-1/2}[\sqrt{T}\bar{g}_T(\theta^0) + G^*\mu_{T,\theta}]\right) + o_p(1). \end{aligned}$$

Hence, the result of part (a) follows by recalling that $R^*\mu_{T,\theta} \xrightarrow{P} \mu_\beta$ [see below (17)] and noting that:

$$\begin{aligned} & P\left(V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)V^{-1/2}G^*\mu_{T,\theta} \\ = \; & V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\left(R^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)^{-1}R^*(G^{*'}V^{-1}G^*)^{-1}G^{*'}V^{-1/2'}V^{-1/2}G^*\mu_{T,\theta} \\ = \; & V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\left(R^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)^{-1}R^*\mu_{T,\theta} \\ = \; & V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\left(R^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)^{-1}\mu_\beta + o_p(1). \end{aligned}$$

Importantly, the same probability limit holds for all θ_T satisfying (17), and hence the proof.
(b) Define $\nu := V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\left(R^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)^{-1}\mu_\beta$. From (a), now it follows by N2 and the full column-rank of $V^{-1/2}G^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}$ [see N8, (25) and (28)] that $LM_T(\theta_T) \xrightarrow{d} \chi^2_{d_R}$ with non-centrality parameter given by $\nu'\nu = \mu_\beta'\left(R^*(G^{*'}V^{-1}G^*)^{-1}R^{*'}\right)^{-1}\mu_\beta$. ■

Proof of Lemma 6: Define the sequence $\{\gamma_T^\dagger : T \geq 1\}$ such that:

$$\gamma_T^\dagger := \arg \inf_{\gamma_0 \in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0', \gamma_0')'\right).$$

By condition (20) on $CI_T(\gamma_S;\epsilon)$, it then follows that $\gamma_T^\dagger$ gives $\theta_T^\dagger = R_S^1 r_0 + S_S^1\gamma_T^\dagger$, for which $\sqrt{T}D_{T,\rho_\theta}^{-1}\Pi_{\rho_\theta}'(\theta_T^\dagger - \theta^0) = O_p(1)$, i.e., (17) holds since (14) also holds. Therefore, the final result follows by Lemma 5. ■

Proof of Lemma 7: The lemma defines the supremum in case of an empty $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in a way that allows us to ignore those cases in the sequel, with the caveat from the discussion above the lemma.

Now, the proof follows in three steps. In Step 1 we show that $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ shrinks to γ_S^0 in probability; more precisely, that the distance between γ_S^0 and $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ converges in probability to zero. Using this, in Step 2 we obtain that this rate cannot be slower than $\lambda_{T,1}$. Using this, in Step 3 we obtain the final result. Details, once stated, are not repeated in the subsequent steps.

Step 1: Cauchy-Schwartz inequality gives: $\|\rho(\theta)\| \leq \left\| \left(\frac{\Lambda_T}{\sqrt{T}}\right)^{-1} \right\| \times \left\| \frac{\Lambda_T}{\sqrt{T}} \rho(\theta) \right\|$, i.e.,

$$\left\| \frac{\Lambda_T}{\sqrt{T}} \rho(\theta) \right\| \geq \frac{1}{\left\| \left(\frac{\Lambda_T}{\sqrt{T}}\right)^{-1} \right\|} \|\rho(\theta)\| = \frac{\|\rho(\theta)\|}{\sqrt{T}\sqrt{\sum_{j=1}^{l} \frac{k_j}{\lambda_{T,j}^2}}}$$

[see N3]. Recall that $d_g = \sum_{j=1}^{l} k_j$. Define the sequence $\lambda_T^* = \min\{\lambda_{T,1}, \ldots, \lambda_{T,l}\}$ for $T \geq 1$. Hence, the above and (13) give that: $\left\| \frac{\sqrt{T}}{\lambda_T^*} E_T\left[\bar{g}_T(\theta)\right] \right\| = \left\| \frac{\sqrt{T}}{\lambda_T^*} \left(\frac{\Lambda_T}{\sqrt{T}} \rho(\theta) \right) \right\| \geq \frac{\|\rho(\theta)\|}{\sqrt{d_g}}$. Take any constant $\varpi > 0$. Consider the outside of the open ball of radius ϖ around γ_S^0. Then, it follows by N1 and the above that:

$$\liminf_T \inf_{\beta \in \mathcal{B}, \gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} \left\| \frac{\sqrt{T}}{\lambda_T^*} E_T[\bar{g}_T(R_S^1 \beta + S_S^1 \gamma)] \right\| \geq \inf_{\beta \in \mathcal{B}, \gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} \frac{\|\rho(\theta)\|}{\sqrt{d_g}} > 0.$$

Taken together with N8 ($\inf_{\theta \in \Theta} \min[\text{eigen values}(V^{-1}(\theta))] > 0$), this gives:

$$\liminf_T \inf_{\beta \in \mathcal{B}, \gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} \left\| V^{-1/2}(R_S^1 \beta + S_S^1 \gamma) \frac{\sqrt{T}}{\lambda_T^*} E_T[\bar{g}_T(R_S^1 \beta + S_S^1 \gamma)] \right\| > 0.$$

Now, note that:

$$V^{-1/2}(\theta) \frac{\sqrt{T}}{\lambda_T^*} \bar{g}_T(\theta) = V^{-1/2}(\theta) \frac{\sqrt{T}}{\lambda_T^*} \left(\bar{g}_T(\theta) - E_T[\bar{g}_T(\theta)] \right) + V^{-1/2}(\theta) \frac{\sqrt{T}}{\lambda_T^*} E_T[\bar{g}_T(\theta)].$$

By N2, N3 and N8, the first term on the RHS is $o_p(1)$ uniformly in $\theta \in \Theta$. Therefore, by using the uniform consistency of $\widehat{V}_T^{-1}(\theta)$ for $V^{-1}(\theta)$ from N8, and the definition of $Q_T(\theta)$ from (11), it follows that

$$\lim_T Pr_T \left(\inf_{\beta \in \mathcal{B}, \gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} (\lambda_T^*)^{-2} \times T \times Q_T(R_S^1 \beta + S_S^1 \gamma) > c \text{ for some } c > 0 \right) = 1,$$

$$\text{and hence,} \quad \lim_T Pr_T \left(\inf_{\beta \in \mathcal{B}, \gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} T \times Q_T(R_S^1 \beta + S_S^1 \gamma) > \bar{c} \text{ for all } \bar{c} < \infty \right) = 1,$$

$$\text{and hence,} \quad \lim_T Pr_T \left(\inf_{\gamma \in \Gamma_S : \|\gamma - \gamma_S^0\| \geq \varpi} T \times Q_T(R_S^1 r_0 + S_S^1 \gamma) > \bar{c} \text{ for all } \bar{c} < \infty \right) = 1. \tag{42}$$

The second line follows since $\liminf_T \lambda_T^* = \infty$ by N3. The third line follows since $r_0 \in \mathcal{B}$ for large T.

Since $\varpi > 0$ is arbitrary, by the definition of $CI_T^{SW}(\gamma_S; r_0, \epsilon)$ in (10) where the critical value is a fixed, finite positive number for a given $\epsilon < 1-\alpha$, it follows from (42) that:[16]

$$\sup_{\gamma_0 \in CI_T^{SW}(\gamma_S; r_0, \epsilon)} \|\gamma_0 - \gamma_S^0\| = o_p(1).$$

Step 2: Take any constant $\varpi > 0$. Define $\{\Gamma_T(\varpi) : T \geq 1\}$, shrinking at rate slower than $\lambda_{T,1}$, as:

$$\begin{aligned}\Gamma_T(\varpi) \quad := \quad & \{\gamma \in \Gamma_S : a_T\|\gamma - \gamma_S^0\| \leq \varpi \text{ and } \lambda_{T,1}\|\gamma - \gamma_S^0\| \geq b_T \text{ for some positive sequences} \\ & \{a_T : T \geq 1\} \text{ and } \{b_T : T \geq 1\} \text{ with } a_T \to \infty, b_T \to \infty \text{ as } T \to \infty\}.\end{aligned}$$

Consider a sequence $\{\gamma_T : T \geq 1\}$ such that $\gamma_T = \arg\inf_{\gamma \in \Gamma_T(\varpi)} T \times Q_T(R_S^1 r_0 + S_S^1 \gamma)$ for each $T \geq 1$. (The sequence need not be unique.) Hence, $\|\gamma_T - \gamma_S^0\| = o(1)$ (although $\lim_T \lambda_{T,1}\|\gamma_T - \gamma_S^0\| = \infty$). Also, $\lim_T \lambda_{T,1}\|r_0 - \beta^0\| < \infty$ by N3 and (14). Therefore, $\theta_T := R_S^1 r_0 + S_S^1 \gamma_T \in \mathcal{N}(\theta^0)$ (as in N4) for large T. This gives, by a mean value expansion of $\rho(\theta_T)$ around $\rho(\theta^0)$ with mean value $\bar{\theta}_T$ (element by element), that:

$$\lambda_{T,1}\Lambda_T^{-1}\sqrt{T}E_T[\bar{g}_T(\theta_T)] = 0 + \left[\rho_\theta(\bar{\theta}_T)R_S^1\right]\lambda_{T,1}(r_0 - \beta^0) + \left[\rho_\theta(\bar{\theta}_T)S_S^1\right]\lambda_{T,1}(\gamma_T - \gamma_S^0)$$

by using N1. N4 and Lemma 10 imply that the terms inside the squared brackets on the RHS are full column-rank. Hence, the second term on the RHS is $O(1)$ whereas the third term diverges to $\pm\infty$. Since $\liminf_T \Lambda_T/\lambda_{T,1} > 0$ by N3, it follows that $\lim_T \sqrt{T}\|E_T[\bar{g}_T(\theta_T)]\| = \infty$. Hence, using the definitions of γ_T and $\Gamma_T(\varpi)$ in conjunction with the result of Step 1, the same arguments as in Step 1 now give:

$$\sup_{\gamma_0 \in CI_T^{SW}(\gamma_S; r_0, \epsilon)} \lambda_{T,1}\|\gamma_0 - \gamma_S^0\| = O_p(1).$$

Step 3: Equipped with the result from Step 2, now we further refine this rate as follows. Define

$$\begin{aligned}\Gamma_T \quad := \quad & \{\gamma \in \Gamma_S : \lambda_{T,1}\|\gamma - \gamma_S^0\| < b_T \text{ for any positive sequence } \{b_T : T \geq 1\} \text{ with } b_T \to \infty, \\ & \|\sqrt{T}D_{T,\rho_\theta}^{-1}\Pi_{\rho_\theta}'(R_S^1(r_0 - \beta^0) + S_S^1(\gamma - \gamma_S^0))\| \geq a_T \text{ for some positive sequence} \\ & \{a_T : T \geq 1\} \text{ with } a_T \to \infty, \text{ and where } r_0 \text{ is as defined in (14)}\}.\end{aligned}$$

Consider a sequence $\{\gamma_T : T \geq 1\}$ such that $\gamma_T = \arg\inf_{\gamma \in \Gamma_T} T \times Q_T(R_S^1 r_0 + S_S^1 \gamma)$ for each $T \geq 1$. Either $\gamma_T \notin \Gamma_T$, in which case Step 2 gives the result, or $\gamma_T \in \Gamma_T$ and hence $\lim_T \lambda_{T,1}\|\gamma_T - \gamma_S^0\| < \infty$.

[16]This argument is used at the end of all the three steps in the proof of this lemma. A rigorous version of this argument is presented in the proof of Lemma 13.2 in Andrews (2017). We focus on establishing an appropriate (for us) version of what Andrews (2017) assumes as the global strong-identification condition for the nuisance parameter. See footnote 6.

Also, $\lim_T \lambda_{T,1}\|r_0 - \beta^0\| < \infty$ by N3 and (14). Therefore, for $\theta_T := R_S^1 r_0 + S_S^1 \gamma_T$, it follows that $\lim \lambda_{T,1}\|\theta_T - \theta^0\| < \infty$, giving $\theta_T \in \mathcal{N}(\theta^0)$ (as in N4) for large T. Furthermore, since $\mu_\beta \neq 0$, $D_{T,\rho\theta}^{-1}$ and $\Pi'_{\rho\theta}$ are nonsingular, and R_S^1 and S_S^1 are full column-rank, it follows that $\eta_T := \sqrt{T} D_{T,\rho\theta}^{-1}\Pi'_{\rho\theta}(\theta - \theta^0) \neq 0$ and, by the definition of Γ_T, that $\lim_T \|\eta_T\| = \infty$.

To proceed, first we use (13) and N1, and by a mean value expansion of $\rho(\theta_T)$ around $\rho(\theta^0)$ with mean value $\bar{\theta}_T$ (element by element), we obtain that:

$$\begin{aligned}
\sqrt{T}\bar{g}_T(\theta_T) &= \sqrt{T}\left(\bar{g}_T(\theta_T) - E_T[\bar{g}_T(\theta_T)]\right) + \Lambda_T\left\{\rho(\theta^0) + \rho_\theta(\theta^0)(\theta_T - \theta^0) + [\rho_\theta(\bar{\theta}_T) - \rho_\theta(\theta^0)](\theta_T - \theta^0)\right\} \\
&= \psi_T(\theta_T) + \left(\frac{\Lambda_T}{\sqrt{T}}\rho_\theta(\theta^0)\Pi_{\rho\theta} D_{T,\rho\theta}\right)\eta_T + \phi_T(\theta_T)
\end{aligned}$$

using $\rho(\theta^0) = 0$ [see N1], and where $\psi_T(\theta_T)$, as defined in N2, is $O_p(1)$ using N8, while

$$\phi_T(\theta_T) := \Lambda_T[\rho_\theta(\bar{\theta}_T) - \rho_\theta(\theta^0)](\theta_T - \theta^0) = O_p\left(\frac{\lambda_{T,l}}{\lambda_{T,1}^2}\right) = o_p(1).$$

The first equality for $\phi_T(\theta)$ follows since $\rho(\theta)$ is twice continuous differentiable in $\mathcal{N}(\theta^0)$ [see N7(a)], and using N3 and the fact that $\lim \lambda_{T,1}\|\theta_T - \theta^0\| < \infty$ (noted above). The second one uses N7(b).[17] Hence:

$$\frac{\sqrt{T}}{\|\eta_T\|}\bar{g}_T(\theta_T) = \frac{1}{\|\eta_T\|}\psi_T(\theta_T) + G^*\frac{\eta_T}{\|\eta_T\|} + o_p\left(\frac{1}{\|\eta_T\|}\right).$$

Hence, as in Step 1 and 2, by using N8, N1, the finiteness and full column-rank of G^* [see (25)], and that $\eta_T \neq 0$ while $\lim_T \eta_T = \infty$, it follows that $T \times Q_T(\theta_T)$ diverges to ∞ in probability. Therefore, using the definitions of γ_T, θ_T and Γ_T in conjunction with the result of Step 2, the same arguments as in Step 1 now give the final result of the lemma:

$$\sup_{\gamma_0 \in CI_T^{SW}(\gamma_S; r_0, \epsilon)} \sqrt{T}\left\| D_{T,\rho\theta}^{-1}\Pi'_{\rho\theta}\left((R_S^1(r_0 - \beta^0) + S_S^1(\gamma_0 - \gamma_S^0)\right)\right\| = O_p(1). \ \blacksquare$$

Proof of Proposition 8: The proof is omitted since it is exactly same as that of Theorem 3.2(ii). $\blacksquare$

[17] Recall from Remark 12 that under the Stock and Wright (2000) setup, $\tilde{\Pi}_{\tilde{\rho}\theta} = I_{d_\theta}$ and $\tilde{D}_{T,\tilde{\rho}\theta} = \sqrt{T}\tilde{\Lambda}_T^{-1}$. Hence, $\eta_T = \tilde{\Lambda}_T(\theta_T - \theta^0)$. Hence, similar to above, an expansion of $\sqrt{T}\bar{g}_T(\theta_T)$ under Stock and Wright (2000)'s setup gives:

$$\sqrt{T}\bar{g}_T(\theta_T) = \psi_T(\theta_T) + \left(\tilde{\rho}_\theta(\theta^0)\frac{\tilde{\Lambda}_T}{\sqrt{T}}\tilde{\Pi}_{\rho\theta}\tilde{D}_{T,\rho\theta}\right)\eta_T + \phi_T(\theta_T) = \psi_T(\theta_T) + \tilde{\rho}_\theta(\theta^0)\eta_T + \phi_T(\theta_T)$$

where, using the same arguments as in the main text but *without using N7(b)*, it follows that:

$$\phi_T(\theta_T) := [\tilde{\rho}_\theta(\bar{\theta}_T) - \tilde{\rho}_\theta(\theta^0)]\tilde{\Lambda}_T(\theta_T - \theta^0) = O_p\left(\frac{1}{\tilde{\lambda}_{T,1}}\right)\eta_T, \text{ i.e., } \|\phi_T(\theta_T)\| = O_p\left(\frac{\|\eta_T\|}{\tilde{\lambda}_{T,1}}\right).$$

Therefore, $\frac{\sqrt{T}}{\|\eta_T\|}\bar{g}_T(\theta_T) = \frac{1}{\|\eta_T\|}\psi_T(\theta_T) + G^*\frac{\eta_T}{\|\eta_T\|} + O_p\left(\frac{1}{\tilde{\lambda}_{T,1}}\right)$. Since weak identification is not allowed, i.e., $\tilde{\lambda}_{T,1} \to \infty$, and since G^* is full column-rank, this means that the rest of the steps in the main text of the proof can follow without change.

Supplemental Appendix D: Unrestricted-by-H_0 plug-in is not advisable

In reference to the key feature (F3) in Section 2, we conduct the same simulation study as in Section 4.2.3. However, instead of the standard plug-in tests, now we consider the unrestricted version of the plug-in tests that replace γ_S in $LM_T(r_0, \gamma_S)$ by its unrestricted-by-H_0 CU-GMM estimator $\widetilde{\gamma}_S := S\widetilde{\theta}$, where $\widetilde{\theta} := \arg\min_{\theta\in\Theta} Q_T(\theta)$ and $Q_T(.)$ is as defined in (11). Here, $\widetilde{\gamma}_S$ is the unrestricted LIML estimator.

Plots similar to Figure 1 are now reported in Figure 2.

As evident from Figure 2, the intuition in (F3) is confirmed by the simulation results for the unrestricted plug-in tests. These tests are no longer invariant to S, and now behave in a way that resembles the behavior of the corresponding S-dependent infeasible tests when H_0 is false. This means that the unrestricted plug-in test has better power when the γ_S with worse identification strength is used for the nuisance parameter. However, this comes at the cost of severe over-rejection of the true H_0 under specifications (ii) and (iii), where the standard plug-in test actually did not display over-rejection in Section 4.2.3 [see Figure 1]. To be clear, neither specification falls under the scope of our results, although Theorem 6 of Guggenberger and Smith (2005) indicates that the standard plug-in test (as in Section 4.2.3) would have had correct asymptotic size under specification (iii), i.e., when $\lambda_{T,1} = 1$ and $\lambda_{T,2} = \sqrt{T}$ and hence the nuisance parameters are strongly identified.

To avoid clutter, we do not report in Figure 2 the results for the unrestricted version of the two-step projection test. (They are available from the author.) But it should be mentioned that we find its empirical power to be extremely poor except under specification (vi) and, to a small extent, under specification (v). The problem of poor power can be roughly attributed to the all too frequent occurrences of very large and even unbounded[18] first-step confidence set when the sample size T is small. It happens because this confidence set does not take advantage of the false H_0 by imposing it. The problem of poor power does not go away even in simulations with very large T, under the three specifications (i)-(iii) that involve at least one weakly identified component (and hence, outside the scope of Section 4.2).

However, we do find that this problem of poor power essentially disappears if, e.g., $T = 2000$, for all the three specifications (iv)-(vi) that are under the scope of our results in Section 4.2. In other words, under these three specifications (iv)-(vi), as in Figure 1, the two-step test behaves similar to the less powerful infeasible and plug-in tests that use the better identified γ_S^0 as the nuisance parameter. Notwithstanding, the intuition from (F3) is not applicable to the two-step test since it may remain invariant to S,[19] and, therefore, resembling the corresponding S-dependent infeasible test is very unlikely. In summary, as noted in (F3), we do not recommend this strategy for the two-step projection test.

[18]Since this a linear model, following convention, we did not impose compactness of Θ (or, $\mathcal{B}$ and Γ_S) in our computation.
[19]This happens if $\gamma_0 := \arg\inf_{\gamma\in CI_T(\gamma_S;\epsilon)} LM_T\left(A_S^{-1}(r_0', \gamma')'\right)$ and r_0 are such that $T \times Q_T(A_S^{-1}(r_0', \gamma_0')') \le \chi^2_{d_g}(1-\epsilon)$.

Figure 2: Empirical rejection probabilities of the infeasible test (infeas) in (8), and the unrestricted version of the plug-in test (LIML (unres)) based on the unrestricted CU-GMM (in this case, unrestricted LIML) estimator for γ_S. Two choices of γ_S, i.e., $\gamma_S = \theta_1$ and $\gamma_S = \theta_2$, are employed for the infeasible test and the unrestricted version of the plug-in test. For all tests, we take $\alpha = .045$. Results are based on 10,000 Monte Carlo trials. Horizontal axis: deviation of H_0 in (2) from truth [also see (14)]. Title: Identification strength that corresponds to specifications (i)-(vi) respectively.