Indirect Inference with Endogenously Missing Exogenous Variables *

Saraswata Chaudhuri[†], David T. Frazier[‡] and Eric Renault[§]

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Abstract

We consider consistent estimation of parameters in a structural model by Indirect Inference (II) when the exogenous variables can be missing at random (MAR) endogenously. We demonstrate that II procedures that simply discard sample units with missing observations can yield inconsistent estimates of the true structural parameters. By inverse probability weighting (IPW) the "complete case" observations, i.e., sample units with no missing variables for the observed and simulated samples, we propose a new method of II to consistently estimate the structural parameters of interest. Asymptotic properties of the new estimator are discussed. We consider a multinomial probit model to illustrate this approach and subsequently consider simulation studies in a variety of discrete choice models with and without dynamics in terms of lagged dependent variables and serially correlated errors. The simulation results demonstrate the severe bias incurred by existing II estimators, and its correction by our new II estimator.

Keywords: Indirect Inference; Missing at Random; Inverse Probability Weighting; Discrete Choice Models.

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[†]McGill University and Cireq, Canada. Email: saraswata.chaudhuri@mcgill.ca.

[‡]Monash University, Australia. Email: david.frazier@monash.edu.

[§]Brown University, United States. Email: eric_renault@brown.edu

1 Introduction

Since the seminal work of Smith (1990, 1993), Gourieroux et al. (1993) and Gallant and Tauchen (1996), Indirect Inference (II) has been used for estimation in a variety of structural models where direct computation of likelihood functions is difficult or intractable. Regardless of the complexity of the underlying structural model, so long as endogenous variables Y can be easily simulated conditional on observed values of exogenous covariates X and Z, II can be used to conduct inference on the structural parameters governing the model under study.

To fix ideas, assume the researcher is interested in conducting inference on the structural parameters $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$ defined by some structural model

$$Y = r(Z, X, \varepsilon; \theta),\ \varepsilon \sim f_\varepsilon,$$

where $r(\cdot)$ is a vector-valued function known up to $\theta \in \Theta$, X and Z are independent of ε and treated as exogenous covariates. II is primarily motivated by settings where the conditional density $f_{Y|X,Z}(\cdot|x, z; \theta)$ renders likelihood-based estimation of θ infeasible or prohibitively difficult, but pseudo-errors $\tilde{\varepsilon} \sim f_\varepsilon$, and hence pseudo-data for Y, can easily be simulated. The rapidity with which pseudo-data can often be simulated, and it's ease of implementation, has led to the use of II in many economic models where likelihood-based estimation is too computationally demanding; see, e.g., the interesting examples in Li (2010) and Bruins et al. (2016), both of which fall under the scope of this paper. To this end, throughout the remainder we set our focus on II-based estimation and inference of structural parameters $\theta \in \Theta$ defined by the above model.

Given the structure of $f_{Y|X,Z}(\cdot|x, z; \theta)$ it is clear that the II simulation step is conducted conditional on observed values of the exogenous covariates $X = x, Z = z$. Therefore, for $\{Y_i, Z_i, X_i\}_{i=1}^N$ denoting an observed sample from (Y, Z, X), if the exogenous covariates have missing units in the sample, say, we only observe some subset of $\{X_i\}_{i=1}^N$, it is not clear how or even if one should simulate endogenous variables when the corresponding units of the exogenous covariates are not observed. As noted by Jiang and Turnbull (2004) (Section 3.4), if sample units are missing and if the data is not "Missing Completely At Random" (MCAR), also known as exogenous missingness/selection, the key tool for II, namely the binding function, may be impossible to infer from simulations. However, missingness patterns that are not completely random are common-place in practice; see, e.g., Wooldridge (2007) for a comprehensive review.

In a similar vein, Altonji et al. (2013) have recently remarked that in some circumstances "accommodating missing data in II is straightforward: after generating a complete set of simulated data, one simply omits observations in the same way they are omitted in the observed data." In this paper we focus on situations where the above argument of Altonji et al. (2013) is invalid due to the impossibility of simulating data that properly mimics the actual missing data mechanism in the data generating process due to missing data in a subset of conditioning variables.

More precisely, our focus is settings where the pattern of missingness is confined to sample units of the exogenous covariates X only, which are needed in order for II to simulate endogenous variables. In contrast to the MCAR assumption, we focus on the case where the covariates are missing endogenously from the sample. Herein, an endogenous pattern of missingness is understood to be the same as "endogenous selection" described in Wooldridge (2007) (pg 1283): we say that the covariates are missing endogenously if the event of missingness in X is related to the endogenous variables Y and possibly the covariates Z. More specifically, we assume throughout that the exogenous covariates in our observed sample are "missing at random"

(MAR). Following Rubin (1976), data are MAR if "the conditional probability of the observed pattern of missing data, given the missing data and the value of the observed data, is the same for all possible values of the missing data."

The MAR assumption covers many empirically relevant situations and has found recent use in economics and econometrics; see, e.g., Hirano et al. (2003), Chen et al. (2005), Wooldridge (2007), Chen et al. (2008), Graham et al. (2012) for cases where the missingness pattern is similar to ours, while Cattaneo (2010), Chaudhuri and Guilkey (2016), Chaudhuri (2016), etc. consider more involved patterns of missingness. In the context of direct estimation, i.e., situations where the likelihood or moments can be analytically evaluated, consistent inference can be carried out under the MAR assumption using some variant of the inverse probability weighting (IPW) approach of Horvitz and Thompson (1952); see, also Robins et al. (1994) and Wooldridge (2007).

Unlike existing solutions in direct estimation, the indirect estimation approach requires the simulation of endogenous variables and, in this way, introduces novel issues in how the MAR assumption can be applied. In particular, because II can only simulate data if the sample units of all exogenous variables are observed, the simulation step of II induces a perverse dependence between the simulated endogenous variables and the process governing the missingness. As a direct consequence, and in contrast to the earlier quoted claim of Altonji et al. (2013), when the exogenous covariates are MAR an II approach that simply discards sample units with missingness will not deliver consistent estimators of the structural parameters. (A more precise discussion on this point is detailed in Section two). However, we briefly mention that since II assumes a given structural model for endogenous variables, there is no issue with missingness if only the endogenous variables are missing. Likewise, the issue is non-existent if the missingness in X depends only on the exogenous covariates Z.

To rectify the issues posed by MAR covariates in II, and extend simulation-based inference procedures to such settings, we construct a novel identification argument (see Section two) that uses IPW, along with the MAR assumption and a particular simulation design, to (jointly) identify the parameters of interest. As we demonstrate, by themselves each of these arguments is insufficient to identify the parameters. Using this novel identification strategy, we construct a feasible II estimator using IPW and detail its asymptotic properties under reasonable assumptions. To the best of our knowledge, this paper constitutes the first use of IPW within indirect estimation methods.

The remainder of the paper is organized as follows. Our proposed II strategy with MAR exogenous variables is discussed in Section two. Section three details implementation of this new II strategy, states the asymptotic theory and proposes an alternative implementation of our approach based on the generalized indirect inference (GII) approach originally proposed in Keane and Smith (2005), and elaborated on in Bruins et al. (2016), which is particularly useful in complex discrete choice models. In Section four, we illustrate our new approach in a multinomial probit model similar to Section nine of Gourieroux et al. (1993) and model four of Bruins et al. (2016). However, due to the missing data problem, we carefully revisit identification of the structural parameters. Section four also contains a small scale Monte Carlo experiment illustrating the performance of our approach in a multinomial probit model. Given the non-smoothness of the binding function in the multinomial probit model, we also consider a GII implementation of our approach. The Monte Carlo results provide compelling evidence on the performance of our II strategy and its alternative GII implementation. Section five concludes, and proofs of the theoretical results are collected in Appendix A. Supplementary simulation examples in the confines of dynamic discrete choice models are contained in Appendix B.

2 Indirect Inference with MAR Exogenous Variables

2.1 Setup and Standard Indirect Inference

We are interested in conducting II in settings where the econometrician does not observe a sample $\{Y_i, Z_i, X_i\}_{i=1}^N$ from (Y, Z, X), but only a subset thereof.[1] In particular, we assume the pattern of missingness is characterized by observing only a subsequence of $\{X_i\}_{i=1}^N$. Following common practice, define the binary random variable D_i with $D_i = 1$ when the vector X_i is observed and $D_i = 0$ if X_i is not observed, and with the notion that if $D_i = 0$ then $D_i X_i = 0$. In this case, the econometrician observes data $\{Y_i, Z_i, D_i, D_i X_i\}_{i=1}^N$ drawn from (Y, Z, D, DX). Throughout the remainder we define $W = (Y', Z')'$ for notational brevity.

We assume the pattern of missingness satisfies a Missing at Random (MAR) assumption for the covariates X, hereafter referred to as the MAR-X assumption. In other words, we assume that, almost surely,[2]

$$\Pr[D = 1 \,|W, X] = \Pr[D = 1 \,|W] = p_0(W) > 0. \tag{1}$$

For simplicity we consider a parametric structure for the missingness so that, for some $\gamma^0 \in \text{Int}(\Gamma) \subset \mathbb{R}^{d_\gamma}$, $p_0(W) = p(W; \gamma^0)$. Together with the structural model for Y and the MAR-X assumption for X in equation (1), we restate the model for the purpose of inference as

$$Y = r(Z, X, \varepsilon; \theta^0),\ \varepsilon \sim f_\varepsilon \tag{2}$$

$$D|(W, X) \sim p_0(W) \equiv p(W; \gamma^0). \tag{3}$$

Our interest lies in II-based estimation and inference on the true unknown value $\theta^0 \in \text{Int}(\Theta) \subset \mathbb{R}^{d_\theta}$ of the structural parameters θ. We briefly remark that the above *modeling framework* is similar to that of Wooldridge (2007), among others. In contrast to Wooldridge (2007), our focus is inference on θ^0 in settings where $f_{Y|X,Z}(\cdot|x, z; \theta)$, or expectations based on $f_{Y|X,Z}(\cdot|x, z; \theta)$, are not tractable and simulation-based methods such as II are often used to conduct inference.

II sets the focus on estimation of $\theta^0 \in \Theta \subset \mathbb{R}^{d_\theta}$ through an intermediate or auxiliary statistic that consistently estimates the unknown value β^0 of some auxiliary parameters $\beta \in \mathcal{B} \subset \mathbb{R}^{d_\beta}, d_\beta \geq d_\theta$. For sake of expositional simplicity, we always define β^0 as the unique solution of the just-identified moment conditions[3]

$$E[m(W, X, \beta)] = 0, \tag{4}$$

where $W = (Y', Z')'$ and X are random vectors. The vector function $m(.,.,.)$ is known and of dimension d_β. Under MAR-X in equation (1), sample counterparts of the moment conditions (4) can only be deduced from the "observed" or "complete case" units $\{D_i \cdot m(W_i, X_i, \beta)\}_{i=1}^N$; i.e., when X_i is not observed we can only compute $D_i m(W_i, X_i, \beta)$. Clearly, revisiting (4) as the complete case moment conditions

$$E[Dm(W, X, \beta)] = 0 \tag{5}$$

[1] In what follows, panel data can easily be accommodated at the cost of more involved notations. However, for simplicity we do not pursue this aspect further.

[2] Wooldridge (2007) stresses the relevance of an extension of assumption (1) (see Wooldridge's Assumption 3.1 (iv) p 1283) to also allow some components of the vector W_i to be unobserved whenever $D_i = 0$. In the context of II, this extension will be immaterial as long as the components of W impacted by the missing data mechanism are only endogenous variables.

[3] See Frazier and Renault (2016) for a discussion of II with over-identified moment conditions.

would lead to the textbook issue of selection bias for estimation of β^0. However, since our goal is not direct estimation of β^0, but II through some (possibly inconsistent) estimate of β^0, this bias may be irrelevant.

Given this fact, a reasonable II procedure would be to estimate θ^0 according to the following steps: first, estimate β^0 from the sample analog of equation (5), i.e., by solving, with respect to (hereafter, wrt) β, $0 = \sum_{i=1}^N D_i m(W_i, X_i, \beta)/N$; second, for $\tilde{\varepsilon} \sim f_\varepsilon$ a simulated random variable, simulate values $Y(\theta) = r(Z, X, \tilde{\varepsilon}; \theta)$ from the structural model and estimate the binding function $\theta \mapsto \beta(\theta)$ using a sample analog of the simulated complete case moments

$$E[Dm(Y(\theta), Z, X, \beta)] = 0, \tag{6}$$

i.e., by solving, wrt β for fixed $\theta \in \Theta$, $0 = \sum_{i=1}^N D_i m(Y_i(\theta), Z_i, X_i, \beta)/N$; third, minimize, wrt θ, the norm difference between the observed and simulated estimators. We call such an estimator the "complete case" II estimator.

The complete case II estimation strategy amounts to a standard application of II using only the complete case sample units $\{D_i Y_i, D_i Z_i, D_i X_i\}_{i=1}^N$, and is incapable of identifying θ^0 in general. To understand why, note that identification would require that the joint distributions of (D, Y, Z, X) and $(D, Y(\theta^0), Z, X)$ be identical.[4] However, this can not be true for the following reason: the simulated error $\tilde{\varepsilon}$ used to generate $Y(\theta^0)$ is, by construction, independent of D, whereas the MAR-X assumption *does not* demand independence between ε (structural error in Y-equation) and D, either unconditionally or conditional on X and Z. Hence, unless D is independent of ε, which rules out endogenous missingness of X, one can not identify θ^0 following the above approach except by happenstance. To further clarify this identification failure we refer the interested reader to Section 4.1 for a simple example illustrating this failure.[5]

2.2 II Based on IPW Under MAR-X

To understand how to construct an II procedure that identifies θ^0 under MAR-X, first consider consistent estimation of β^0 defined by equation (4). Using $p_0(W)$ and the MAR-X assumption in (1), the following simple argument allows us to correctly identify the auxiliary parameters β^0 since, for all $\beta \in \mathcal{B}$

$$E\left[\frac{D}{p_0(W)} m(Y, Z, X, \beta)\right] = E\left[E\left[\frac{D}{p_0(W)}\middle| W, X\right] m(Y, Z, X, \beta)\right] = E\left[m(Y, Z, X, \beta)\right]. \tag{7}$$

As noted by Jiang and Turnbull (2004), in the case of missing data, the key property for performing II using the auxiliary equation (7) is to ensure that the resulting binding function identifies θ^0. This subsequently requires that, for all $\beta \in \mathcal{B}$, $\theta \in \Theta$,

$$E\left[\frac{D}{p_0(W)} m(Y(\theta), Z, X, \beta)\right] = E\left[m(Y(\theta), Z, X, \beta)\right]. \tag{8}$$

Demonstrating satisfaction of (8) requires a more precise study of the expectations used above. In equation (7) the notations are straightforward: expectations are computed with

[4] At a minimum, identification requires that both distributions deliver identical expectations in equations (5) and (6) for any β.

[5] Interestingly enough, this hurdle may actually be overcome by considering joint simulation from D, Y. Since such an approach would require simulation from the missingness mechanism, we leave this approach for further study (see Chaudhuri et al., 2016).

respect to the joint distribution of (D, Y, Z, X) given by the data generating process (DGP). The expectation operator in (8) involves jointly the generating processes for the observed and simulated data. To highlight this difference, we analyze each case in turn, starting with the observed data.

The observed data $\{D_i, Y_i, Z_i, D_iX_i\}_{i=1}^N$, where $D_iX_i = 0$ if $D_i = 0$ and X_i else, can be seen as the output of the following mechanism:
(O1) Exogenous variables $\{Z_i, X_i\}_{i=1}^N$, possibly partially latent, are generated by a completely unknown DGP.
(O2) Stochastic errors $\{\varepsilon_i\}_{i=1}^N$ are drawn i.i.d. from the known probability distribution of ε, with all draws independent of $\{Z_i, X_i\}_{i=1}^N$.
(O3) Endogenous variables $\{Y_i\}_{i=1}^N$ are observed as a result of the DGP: $Y_i = r(Z_i, X_i, \varepsilon_i; \theta^0)$, with θ^0 the true unknown value of the structural parameters.
(O4) $\{D_i\}_{i=1}^N$ is drawn in the product of conditional distributions of D_i given $\{Y_i, Z_i, X_i\}_{i=1}^N$. Moreover, these conditional distributions, for $i = 1, ..., N$, are assumed (by MAR-X) not to depend on X_i.

A similar procedure is implicitly considered when simulating the endogenous variables. However, unlike step **(O2)** above, for some integer $S \geq 1$ we draw $s = 1, ..., S$ independent simulated samples of i.i.d. errors $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$, where $\tilde{\varepsilon}_{is} \sim f_\varepsilon$, and $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$ is independent of $\{\varepsilon_i, Z_i, D_iX_i, D_i\}_{i=1}^N$ by construction. Given $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$, and in accordance with **(O3)** above, we wish to simulate, for any $\theta \in \Theta$,

$$Y_{is}(\theta) = r(Z_i, X_i, \tilde{\varepsilon}_{is}; \theta).$$

While $Y_{is}(\theta)$ can always be defined, it is not feasible to simulate $Y_{is}(\theta)$ when $D_i = 0$. Therefore, in place of $Y_{is}(\theta)$ we instead consider the feasible simulator

$$Y_{is}^D(\theta) = \begin{cases} Y_{is}(\theta) & \text{if } D_i = 1 \\ 0 & \text{if } D_i = 0 \end{cases}$$

The s-th simulation step produces a sequence $\{Z_i, D_iX_i, \varepsilon_i, D_i, \tilde{\varepsilon}_{is}\}_{i=1}^N$ of i.i.d. draws in a joint distribution that defines, through known transformations, the joint distribution of the variables at stake to compute the expectation in (8).

Note that the simulated endogenous variables $Y_{is}^D(\theta)$ directly depend on D_i, whereas the observed endogenous variables Y_i do not. Therefore, the distribution of $(Y_{is}^D(\theta), D_i)$ will not exactly replicate the distribution of (Y_i, D_i) even when $\theta = \theta^0$; i.e., because of the MAR-X covariates, we are forced to simulate $Y_{is}^D(\theta)$ according to a (slightly) misspecified structural model. Moreover, as explained in the last paragraph in Section 2.1, it is precisely this perverse dependence between $Y_{is}^D(\theta)$ and D_i that ensures complete case II will not identify θ^0 in general.

However, note that because $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$ is independent of $\{\varepsilon_i, Z_i, X_i, D_i\}_{i=1}^N$, it is also independent of the missing data mechanism encapsulated by $\{D_i\}_{i=1}^N$, a feature that is not replicated by the actual errors $\{\varepsilon_i\}_{i=1}^N$. Therefore, D_i is endowed with an exogeneity status in regards to the *simulated errors* $\tilde{\varepsilon}_{is}$. In particular, because D_i is independent of $\tilde{\varepsilon}_{is}$, for each $s = 1, ..., S$, given $(\varepsilon_i, Z_i, X_i)$ since $\tilde{\varepsilon}_{is}$ is jointly independent of $(\varepsilon_i, Z_i, X_i, D_i)$, we have that

$$D_i \perp Y_{is}^D(\theta) \;\big|\; W_i, X_i. \tag{9}$$

With the conditional independence $D_i \perp Y_{is}^D(\theta) \;\big|\; W_i, X_i$ generated through the simulation

step, the validity of equation (8) follows from the following arguments:

$$\begin{aligned}
E\left[\frac{D}{p_0(W)}m(Y^D(\theta),Z,DX,\beta)\right] &= E\left[\frac{D}{p_0(W)}m(Y(\theta),Z,X,\beta)\right] \text{ (by definition of } D) \\
&= E\left[E\left[\frac{D}{p_0(W)}m(Y(\theta),Z,X,\beta)\middle|W,X\right]\right] \text{ (by L.I.E.)} \\
&= E\left[E\left[\frac{D}{p_0(W)}\middle|W,X\right]E\left[m(Y(\theta),Z,X,\beta)|W,X\right]\right] \text{ (by (9))} \\
&= E\left[E\left[\frac{D}{p_0(W)}\middle|W\right]E\left[m(Y(\theta),Z,X,\beta)|W,X\right]\right] \text{ (by MAR-X)} \\
&= E\left[m(Y(\theta),Z,X,\beta)\right] \text{ (by definition of } p_0(W)). \qquad (10)
\end{aligned}$$

Equation (10) is precisely what we require to construct a feasible II approach that identifies θ^0, when based on the auxiliary model (4), in spite of the missing data problem.[6]

More precisely, for $\beta^0 \in \mathcal{B}$ and $\theta \mapsto \beta(\theta)$ defined by, respectively,

$$\begin{aligned}
E\left[m(Y,Z,X,\beta^0)\right] &= 0, \\
E\left[m(Y(\theta),Z,X,\beta(\theta))\right] &= 0,
\end{aligned}$$

under the standard identification assumption $\beta^0 = \beta(\theta) \Leftrightarrow \theta = \theta^0$, comparison of (7) and (8) suggests a feasible II approach with missing data based on the IPW moment conditions

$$E\left[\frac{D}{p_0(W)}m(Y,Z,X,\beta^0)\right] = 0 \qquad (11)$$

$$E\left[\frac{D}{p_0(W)}m(Y(\theta),Z,X,\beta(\theta))\right] = 0. \qquad (12)$$

This is where the novelty of our approach lies. Identification of θ^0, by means of (7) and (8), with equation (8) satisfied by equation (10), does not result directly from the use of IPW and MAR-X in (1), which is the case in direct inference approaches, but also requires the conditional independence introduced through the simulation step.

3 IPW Indirect Inference (IPW-II)

Given that a path to identification of θ^0 is now clear, in this section we discuss precise implementation of an inverse probability weighted II (IPW-II) approach under MAR-X missing data. Asymptotic properties of the ensuing IPW-II approach are discussed in Section 3.4. Section 3.5 presents a computationally friendly implementation of this approach for non-smooth problems using the generalized II method in Bruins et al. (2016).

[6]Note that independence between $\tilde{\varepsilon}$ and (ε, Z, X), as in a standard II context, is insufficient to yield equation (10).

3.1 Estimation of the Auxiliary Model Parameters

3.1.1 Observed Data

Following the identification strategy outlined in Section two, with $W_i = (Y_i', Z_i')'$, define the auxiliary parameter estimate $\widehat{\beta}_N$ as the solution of

$$\frac{1}{N}\sum_{i=1}^{N}\frac{D_i}{p(W_i,\widehat{\gamma}_N)}m(Y_i,Z_i,X_i,\widehat{\beta}_N) = 0,$$

where $\widehat{\gamma}_N$ is the maximum likelihood estimator, the solution to $0 = \sum_{i=1}^{N} l_{i,\gamma}(\gamma)$, with $l_{i,\gamma}(\gamma) = l_\gamma(D_i, W_i, \gamma)$ the score vector of the parametric model describing the missing data mechanism:

$$l_{i,\gamma}(\gamma) = \frac{\partial}{\partial\gamma}\log\left[(p(W_i,\gamma))^{D_i}(1-p(W_i,\gamma))^{1-D_i}\right] = \frac{[D_i - p(W_i,\gamma)]}{p(W_i,\gamma)(1-p(W_i,\gamma))}\frac{\partial p(W_i,\gamma)}{\partial\gamma}.$$

Note that $(\widehat{\beta}_N', \widehat{\gamma}_N')'$ can also be seen as a joint GMM estimator provided by the just identified moment conditions

$$\begin{aligned} E\left[\frac{D_i}{p(W_i;\gamma)}m(Y_i,Z_i,,X_i,\beta)\right] &= 0 \\ E\left[l_\gamma(D_i,W_i,\gamma)\right] &= 0 \end{aligned}$$

It is well known (see, e.g., Breusch et al., 1999, Lemma 1, p93) that we can obtain an asymptotically equivalent GMM estimator by instead considering the moment conditions:

$$\begin{aligned} E\left[m_i^*(\gamma,\beta) - \Pi\left[m_i^*(\gamma,\beta) \mid l_{i,\gamma}(\gamma)\right]\right] &= 0 \\ E\left[l_\gamma(D_i,W_i,\gamma)\right] &= 0 \end{aligned} \tag{13}$$

where $m_i^*(\gamma,\beta) = \frac{D_i}{p(W_i;\gamma)}m(Y_i,Z_i,X_i,\beta)$ and, for $\beta = \beta^0$ and $\gamma = \gamma^0$, $\Pi\left[m_i^*(\gamma^0,\beta^0) \mid l_{i,\gamma}(\gamma^0)\right]$ is the affine population regression of $m_i^*(\gamma^0,\beta^0)$ on $l_{i,\gamma}(\gamma^0)$:

$$\begin{aligned} \Pi\left[m_i^*(\gamma^0,\beta^0) \mid l_{i,\gamma}(\gamma^0)\right] &= \Omega_{12}\Omega_{22}^{-1}l_{i,\gamma}(\gamma^0), \\ \Omega_{12} &= \mathrm{Cov}\left[m_i^*(\gamma^0,\beta^0), l_{i,\gamma}(\gamma^0)\right], \quad \Omega_{22} = \mathrm{Var}\left[l_{i,\gamma}(\gamma^0)\right]. \end{aligned}$$

Clearly, the two moments in (13) are uncorrelated at γ^0, β^0, allowing us to compute directly the asymptotic distribution of the GMM estimator $\widehat{\beta}_N$ from its asymptotic expansion[7]

$$\begin{aligned} \sqrt{N}\left(\widehat{\beta}_N - \beta^0\right) &= -\left[G_0'V_0^{-1}G_0\right]^{-1}G_0'V_0^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{m_i^*(\gamma^0,\beta^0) - \Pi\left[m_i^*(\gamma^0,\beta^0) \mid l_{i,\gamma}(\gamma^0)\right]\right\} + o_P(1) \\ &= -G_0^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{m_i^*(\gamma^0,\beta^0) - \Pi\left[m_i^*(\gamma^0,\beta^0) \mid l_{i,\gamma}(\gamma^0)\right]\right\} + o_P(1), \end{aligned}$$

where

$$G_0 = E\left[\frac{D_i}{p(W_i;\gamma^0)}\frac{\partial m(W_i,X_i,\beta^0)}{\partial\beta'}\right], \tag{14}$$

$$V_0 = \mathrm{Var}\left[m_i^*(\gamma^0,\beta^0)\right] - \mathrm{Var}\left[\Pi\left[m_i^*(\gamma^0,\beta^0) \mid l_{i,\gamma}(\gamma^0)\right]\right]. \tag{15}$$

[7] Precise regularity conditions ensuring the validity of this expansion are given as Assumptions **A1**-**A5** in the appendix.

Remarks:

(1) Applying again the MAR-X property to G_0 yields

$$G_0 \;=\; E\left[E\left[\frac{D_i}{p(W_i;\gamma^0)}\middle|W_i,X_i\right]\frac{\partial m(W_i,X_i,\beta^0)}{\partial\beta'}\right] \;=\; E\left[\frac{\partial m(W_i,X_i,\beta^0)}{\partial\beta'}\right],$$

as if we had no missing data. However, due to the missing data problem, the formula (14) provides the natural way to estimate G_0 from its sample counterpart after plugging in consistent estimators of γ^0,β^0. Similarly,

$$\mathrm{Var}\left[m_i^*(\gamma^0,\beta^0)\right] \;=\; E\left[m_i^*(\gamma^0,\beta^0)m_i^*(\gamma^0,\beta^0)'\right] \;=\; E\left[\frac{1}{p(W_i;\gamma^0)}m(W_i,X_i,\beta^0)m'(W_i,X_i,\beta^0)\right]$$

should be estimated from the sample counterpart of (15) rather than the above equation. However, the division by $p(W_i;\gamma^0)$ in the above shows the price we pay, in terms of accuracy of $\widehat{\beta}_N$, for the missing data problem.

(2) The asymptotic variance of $\sqrt{N}(\widehat{\beta}_N-\beta^0)$, given by $G_0^{-1}V_0G_0'^{-1}$, is smaller (in terms of comparison of positive semi-definite matrices) than the asymptotic variance of a GMM estimator for β^0 obtained using the true unknown propensity score $p_0(W)=p(W;\gamma^0)$ and moments

$$E\left[\frac{D_i}{p_0(W_i)}m_i(Y,Z,X,\beta)\right]=0, \tag{16}$$

for which the resulting asymptotic variance would be given by $G_0^{-1}\left[\mathrm{Var}\left\{m_i^*(\gamma^0,\beta^0)\right\}\right]G_0'^{-1}$.

This remark is sometimes summarized by a kind of puzzling statement: "it is better to estimate the weights by a conditional MLE procedure than using known weights (if we knew them)" (Wooldridge, 2007). The explanation of this anomalous statement is simple: we take advantage of the moments provided by the score vector $l_{i,\gamma}(\gamma^0)$ to reduce the variance of $m_i^*(\gamma^0,\beta^0)$ by computing the residual of its regression on $l_{i,\gamma}(\gamma^0)$. The possible efficiency loss when using GMM based only on (16) instead of the GMM estimator $\widehat{\beta}_N$ is not due to the knowledge of γ^0 but to the omission of the second set of moment conditions $l_{i,\gamma}(\gamma^0)$.

3.1.2 Simulated Data

For a given integer $S\geq 1$, we draw $s=1,...,S$ independently simulated samples of i.i.d. errors $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$ from the known probability distribution of ε with, for each $s=1,....,S$, $\{\tilde{\varepsilon}_{is}\}_{i=1}^N$ independent of $\{\varepsilon_i,Z_i,X_i,D_i\}_{i=1}^N$. We can then compute $Y_{is}^D(\theta)$ and define the estimator $\tilde{\beta}_{N,s}(\theta)$ as the solution of[8]

$$\sum_{i=1}^N\frac{D_i}{p(W_i,\widehat{\gamma}_N)}m\left(Y_{is}^D(\theta),Z_i,X_i,\tilde{\beta}_{N,s}(\theta)\right)=0.$$

Following similar arguments to those developed in the previous section, when N is large, (see also (19))

$$\sqrt{N}\left(\tilde{\beta}_{N,s}(\theta^0)-\beta^0\right)=-G_0^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^N\left\{m_{is}^*(\gamma^0,\beta^0;\theta^0)-\Pi\left[m_{is}^*(\gamma^0,\beta^0;\theta^0)\;\middle|\;l_{i,\gamma}(\gamma^0)\right]\right\}+o_P(1), \tag{17}$$

[8]For the exact construction of $Y_{is}^D(\theta)$ we refer the reader to Sections 2.2 and 3.3.

where $m^*_{is}(\gamma,\beta;\theta) = \frac{D_i}{p(W_i;\gamma)} m(Y_{is}(\theta), Z_i, X_i, \beta)$,

$$\begin{aligned} \Pi\left[m^*_{is}(\gamma^0,\beta^0;\theta) \mid l_{i,\gamma}(\gamma^0)\right] &= \Omega_{12}(\theta)\Omega_{22}^{-1} l_{i,\gamma}(\gamma^0), \\ \Omega_{12}(\theta) &= \mathrm{Cov}\left[m^*_{is}(\gamma^0,\beta^0;\theta), l_{i,\gamma}(\gamma^0)\right] \end{aligned}$$

Note that $\Omega_{12}(\theta)$ does not depend on (i,s) since all draws of $(Z, X, \tilde{\varepsilon}_{is})$ are drawn in the same distribution, which corresponds to the distribution of $(Z_i, X_i, \varepsilon_i)$. However, it is critical to note that,

$$\Omega_{12}(\theta^0) = \mathrm{Cov}[m^*_{is}(\gamma^0,\beta^0;\theta^0), l_{i,\gamma}(\gamma^0)] \neq \mathrm{Cov}[m^*_i(\gamma^0,\beta^0), l_{i,\gamma}(\gamma^0)] = \Omega_{12},$$

which follows from the fact that, for $\mathcal{D}(\varepsilon_i, Z_i, X_i, D_i)$ the joint probability distribution of $(\varepsilon_i, Z_i, X_i, D_i)$, in general

$$\mathcal{D}(\varepsilon_i, Z_i, X_i, D_i) \neq \mathcal{D}(\tilde{\varepsilon}_i, Z_i, X_i, D_i).$$

This discrepancy is a consequence of the missingness indicator D_i being exogenous with regards to the simulated errors $\tilde{\varepsilon}_{is}$, since $\tilde{\varepsilon}_{is}$ is, by construction, independent of (Z_i, X_i, D_i). In contrast to $\tilde{\varepsilon}_{is}$, MAR-X does not require that the error ε_i be independent of (Z_i, X_i, D_i) since D_i need not be independent of Y_i given (Z_i, X_i).

Following the first II estimator of Gourieroux et al. (1993) (see their Proposition 1 pS89), an estimator of θ^0 can be obtained by calibrating the value of θ in order to minimize the distance, in some norm, between $\widehat{\beta}_N$ and the average value of the simulated auxiliary estimators

$$\bar{\beta}_{N,S}(\theta) = \frac{1}{S}\sum_{s=1}^{S} \tilde{\beta}_{N,s}(\theta).$$

From $\bar{\beta}_{N,S}(\theta)$ and (17), we deduce that, for given fixed S,

$$\sqrt{N}\left(\bar{\beta}_{N,S}(\theta^0) - \beta^0\right) = -G_0^{-1}\frac{\sqrt{N}}{N\cdot S}\sum_{i=1}^{N}\sum_{s=1}^{S}\left\{m^*_{is}(\gamma^0,\beta^0;\theta^0) - \Pi\left[m^*_{is}(\gamma^0,\beta^0;\theta^0) \mid l_{i,\gamma}(\gamma^0)\right]\right\} + o_P(1).$$

3.2 The Calibration Step: Wald-type IPW-II

Given auxiliary parameter estimates $\widehat{\beta}_N$, $\bar{\beta}_{N,S}(\theta)$, a Wald-type IPW-II estimator for θ^0 is obtained as

$$\widehat{\theta}^W_N(\Upsilon) := \arg\min_{\theta\in\Theta}\left[\widehat{\beta}_N - \bar{\beta}_{N,S}(\theta)\right]' \Upsilon_N^{-1}\left[\widehat{\beta}_N - \bar{\beta}_{N,S}(\theta)\right], \tag{18}$$

where Υ_N is a sequence of positive-definite weighting matrices. The notation $\widehat{\theta}^W_N(\Upsilon)$ stresses that the asymptotic distribution of this estimator depends on the choice of Υ_N^{-1}.

Standard arguments for minimum distance estimators tell us that the optimal weighting matrix is to take $\Upsilon_N \xrightarrow{P} \Upsilon(S)$, where

$$\Upsilon(S) := \lim_{N\to\infty} \mathrm{Var}\left\{\sqrt{N}\left(\widehat{\beta}_N - \bar{\beta}_{N,S}(\theta^0)\right)\right\}.$$

To deduce the form of $\Upsilon(S)$, we first use the expansions $\sqrt{N}(\widehat{\beta}_N - \beta^0)$ and $\sqrt{N}(\bar{\beta}_{N,S}(\theta^0) - \beta^0)$, given in the previous subsections, to find

$$\sqrt{N}\left(\bar{\beta}_N - \widehat{\beta}_{N,S}(\theta^0)\right) = -G_0^{-1}\sqrt{N}\left[\bar{\xi}_{N,S} - C_0\Omega_{22}^{-1}\sum_{i=1}^{N} l_{i,\gamma}(\gamma^0)/N\right] + o_P(1),$$

where $C_0 = [\Omega_{12} - \Omega_{12}(\theta^0)]$ and, for $\xi_{i,S} = m_i^*(\gamma^0, \beta^0) - \frac{1}{S}\sum_{s=1}^S m_{is}^*(\gamma^0, \beta^0; \theta^0)$,

$$\bar{\xi}_{N,S} = \frac{1}{N}\sum_{i=1}^N \xi_{i,S} \equiv \frac{1}{N}\sum_{i=1}^N \left[m_i^*(\gamma^0, \beta^0) - \frac{1}{S}\sum_{s=1}^S m_{is}^*(\gamma^0, \beta^0; \theta^0) \right].$$

Noting that,

$$\text{Cov}\left(\sqrt{N}\bar{\xi}_{N,S} \,,\; C_0\Omega_{22}^{-1}\sum_{i=1}^N l_{i,\gamma}(\gamma^0)/\sqrt{N} \right) = C_0\Omega_{22}^{-1}C_0' = \text{Var}\left(C_0\Omega_{22}^{-1}\sum_{i=1}^N l_{i,\gamma}(\gamma^0)/\sqrt{N} \right)$$

and for $W_0(S) = \lim_{N\to\infty} \text{Var}\left\{ \sqrt{N}\bar{\xi}_{N,S} \right\} = E\left[\xi_{i,S} \cdot \xi_{i,S}' \right]$, $\Upsilon(S)$ then has the following form:

$$\Upsilon(S) = G_0^{-1}\left[W_0(S) - C_0\Omega_{22}^{-1}C_0' \right] G_0^{-1\prime}.$$

Remarks:

(1) The term $W_0(S)$ in $\Upsilon(S)$ can further be decomposed by noting the following. One,

$$\begin{aligned} \text{Var}\left[m_{is}^*(\gamma^0, \beta^0; \theta^0) \right] &= E\left[\frac{D_i}{p^2(W_i; \gamma^0)} m(Y_{is}^D(\theta^0), Z_i, X_i, \beta^0) m'(Y_{is}(\theta^0), Z_i, X_i, \beta^0) \right] \\ &= E\left[\frac{1}{p(W_i; \gamma^0)} m(Y_{is}(\theta^0), Z_i, X_i, \beta^0) m'(Y_{is}(\theta^0), Z_i, X_i, \beta^0) \right] = \text{Var}\left[m_i^*(\gamma^0, \beta^0) \right], \end{aligned}$$

where the second equality comes from an argument similar to the one used to prove (10) and the third equality is implied by the fact that the joint distributions satisfy

$$\mathcal{D}\left(\varepsilon_i, Z_i, X_i \right) = \mathcal{D}\left(\tilde{\varepsilon}_{is}, Z_i, X_i \right), \tag{19}$$

with this distributional equivalence being (partly) why simulated and observed expectations can still coincide, such as, e.g., G_0 defined in (14). Two, by the same logic, for $s, s' = 1, ..., S$

$$\text{Cov}\left[m_i^*(\gamma^0, \beta^0), m_{is}^*(\gamma^0, \beta^0; \theta^0) \right] = \text{Cov}\left[m_{is}^*(\gamma^0, \beta^0; \theta^0), m_{is'}^*(\gamma^0, \beta^0; \theta^0) \right].$$

Introducing the notations,

$$I_0 = \text{Var}\left[m_i^*(\gamma^0, \beta^0) \right], \quad K_0 = \text{Cov}\left[m_i^*(\gamma^0, \beta^0), m_{is}^*(\gamma^0, \beta^0; \theta^0) \right],$$

elementary algebra then yields a familiar form (see Gourieroux et al., 1993, pS109):[9]

$$W_0(S) = \left(1 + \frac{1}{S} \right)(I_0 - K_0).$$

(2) The term K_0 can be further decomposed, by noting that, for $s, s' = 1, ..., S$, even if $s = s'$,

$$\begin{aligned} K_0 &= \text{Cov}\left\{ E[m_{is}^*(\gamma^0, \beta^0; \theta^0) \,|Z_i, D_i X_i] \,, E[m_{is'}^*(\gamma^0, \beta^0; \theta^0) \,|Z_i, D_i X_i] \right\} \\ &= \text{Var}\left\{ E[m_{is}^*(\gamma^0, \beta^0; \theta^0) \,|Z_i, D_i X_i] \right\} = \text{Var}\left\{ E[m_i^*(\gamma^0, \beta^0) \,|Z_i, D_i X_i, D_i] \right\} \end{aligned}$$

which yields the following alternative specification for $I_0 - K_0$:

$$I_0 - K_0 = \text{Var}\left\{ m_i^*(\gamma^0, \beta^0) - E[m_i^*(\gamma^0, \beta^0) \,|Z_i, D_i X_i, D_i] \right\}.$$

This expression makes explicit the efficiency gain due to the fact that we have not simulated (Z, X, D).

[9] The term K_0 is non-zero in general because the observed and simulated samples both have in common the exogenous variables X and Z.

3.3 Alternative IPW-II Implementation

It is well known that the different approaches to choosing a metric between $\widehat{\beta}_N$ and $\bar{\beta}_{N,S}(\theta)$ for the purpose of II on θ correspond to the trinity of asymptotic tests. As summarized by Bruins et al. (2015), following a nomenclature "due to Eric Renault, the Wald and LR [likelihood ratio] approaches were first proposed in Smith (1990, 1993) and later extended by GMR. The LM [Lagrange multiplier] approach was first proposed in Gallant and Tauchen (1996)."

While GMR have stressed that the LR approach may imply some efficiency loss, they also show (see their Section 2.5) that, as far as first-order asymptotics are concerned, the family of Wald-II estimators coincides with the family of LM-II estimators. However, due to the fact that our analysis depends on the matrix $C_0 = \Omega_{12} - \Omega_{12}(\theta^0) \neq 0$, in general, the results of GMR may not be directly applicable to our IPW-II estimators.

In addition, GMR consider the LM and LR approaches only in the context where the moment conditions used to estimate the auxiliary parameters are defined by the gradient of some objective function, like a pseudo-score. In such cases, G_0 is a symmetric Hessian matrix, and in some circumstances this Hessian matrix may coincide with the outer product matrix I_0. Since we only consider parameters defined as zeros of just-identified moment conditions, the Jacobian matrix G_0 is nonsingular but G_0 has no reason to coincide with the symmetric, positive-definite matrix I_0.

For a sequence of positive-definite weighting matrices $\Psi_N \xrightarrow{P} \Psi$, with Ψ positive-definite, a general Wald IPW-II estimator $\widehat{\theta}_N^W(\Psi)$ of θ can be defined as the solution to the minimization program:

$$\widehat{\theta}_N^W(\Psi) = \arg\min_{\theta\in\Theta} \left[\widehat{\beta}_N - \bar{\beta}_{N,S}(\theta)\right]' \Psi_N \left[\widehat{\beta}_N - \bar{\beta}_{N,S}(\theta)\right].$$

From Section 3.2, and standard argument for minimum distance estimation, the optimal choice Ψ^* of Ψ is given by $\Psi^* = \Upsilon^{-1}(S) \equiv G_0' H_0^{-1}(S) G_0$, where

$$H_0(S) = W_0(S) - C_0\Omega_{22}^{-1}C_0', \;\; W_0(S) = \left(1 + \frac{1}{S}\right)[I_0 - K_0]$$

In the case where $C_0 = 0$, and G_0 a Hessian matrix, GMR demonstrate that LR-type II estimators are asymptotically equivalent to $\widehat{\theta}_N^W(G_0)$ and not efficient in general since

$$G_0 \neq G_0 \left[I_0 - K_0\right]^{-1} G_0'.$$

The Wald-type II estimator $\widehat{\theta}_N^W(\Psi)$ can be computationally expensive when $\bar{\beta}_{N,S}(\theta)$ is not known in closed form. Even though the moment conditions of the auxiliary model are not necessarily defined as a gradient vector, we can extend the original LM-II approach by defining a LM IPW-II estimator $\widehat{\theta}_N^{LM}(A)$ as the solution of the minimization program:

$$\widehat{\theta}_N^{LM}(A) = \arg\min_{\theta\in\Theta} \left[M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right)\right]' A_N \left[M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right)\right],$$

where A_N is a sequence of positive-definite matrices with probability limit A and

$$M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right) = \frac{1}{N \cdot S} \sum_{i=1}^{N} \sum_{s=1}^{S} \frac{D_i}{p(W_i; \widehat{\gamma}_N)} m(Y_{is}^D(\theta), Z_i, X_i, \widehat{\beta}_N).$$

When $C_0 = 0$, GMR have shown that the estimator $\widehat{\theta}_N^{LM}(A)$ is asymptotically equivalent to $\widehat{\theta}_N^{W}(G_0 A G_0')$. The extension to our more general case is straightforward with the optimal choice A^* of A given by

$$A^* = H_0^{-1}(S).$$

Exact implementation of the LM-type IPW-II approach can be carried out using the following algorithm, which deals with potential non-smoothness, in θ, of $M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right)$.[10]

Algorithm for implementing of IPW-II

- **Step 0:** Using the observed $\{W_i, D_i\}_{i=1}^N$ estimate $\widehat{p}_0(W_i) := p(W_i; \widehat{\gamma}_N)$ for each $i = 1, \ldots, N$ where $\widehat{\gamma}_N$ is a the maximum likelihood estimator based on any *given* parametric specification $p(W; \gamma)$ for $p_0(W)$, and where γ is some $d_\gamma \times 1$ unknown parameter.

- **Step 1:** Using the observed sample $\{W_i, D_i, D_i X_i\}_{i=1}^N$, obtain $\widehat{\beta}_N$ as:
  $\widehat{\beta}_N \;:=\; \arg_{\beta \in \mathcal{B}} \left\{ \frac{1}{N} \sum_{i=1}^N \frac{D_i}{p(W_i;\gamma)} m(Y_i, Z_i, X_i, \beta) = 0 \right\}$,

- **Step 2a:** Sort the observed sample so that the first $N_1 = \sum_{i=1}^N D_i$ units have $D_i = 1$, i.e., have X_i observed. For any given $\theta \in \Theta$, and for each $i = 1, \ldots, N_1$, generate:

  $$\widetilde{\varepsilon}_{is} \overset{\text{i.i.d.}}{\sim} f_\varepsilon, \quad Y_{is}(\theta) = r(Z_i, X_i, \theta, \widetilde{\varepsilon}_{is}) \text{ for } s = 1, \ldots, S$$

  where S is the pre-specified number of simulations. Set $Y_{is}(\theta) = 0$ for $s = 1, \ldots, S$ and $i = N_1 + 1, \ldots, N$. By generating $Y_{is}(\theta)$ in this fashion we conform to the earlier stated definition of $Y_{is}^D(\theta)$ in Section 2.2.

- **Step 2b:** For any given positive definite matrix A_N, obtain the II estimator $\widehat{\theta}_N^{LM}(A)$ as:

  $$\left\| M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N^{LM}(A)\right) \right\|_{A_N} \;\leq\; o_P\left(N^{-1/2}\right) + \inf_{\theta \in \Theta} \left\| M_{N,S}\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right) \right\|_{A_N}. \tag{20}$$

  We call $\widehat{\theta}_N^{LM}(A)$ a LM-type IPW-II estimator of θ^0.

Remarks:

(1) The IPW-II procedure models $p_0(W)$ parametrically and is susceptible to misspecification. Adverse consequences of parametric misspecification of $p_0(W)$ in Step 0, and remedy thereof by using doubly robust estimating functions for β or by nonparametric estimation of $p_0(W)$ have been studied for general direct IPW estimators [e.g., Scharfstein et al. (1999), Hirano et al. (2003), Chen et al. (2008)]. Extensions of these results to indirect estimators is considered in Chaudhuri et al. (2016).

3.4 Asymptotic Distribution of the IPW-II Estimator

We now provide precise results on consistency and asymptotic normality of the IPW-II estimator $\widehat{\theta}_N^{LM}(A)$ in (20).[11] We deviate from the standard II treatment and present results that

[10] Such situations arise in simulation-based estimation of discrete choice models because the simulated dependent variable, as a function of θ, i.e., $Y(\theta)$, can change discretely (e.g. from 0 to 1) with an infinitesimal change in θ.

[11] Equivalent results can be obtained for the Wald-type IPW-II estimator. However, the arguments mirror those in Gourieroux et al. (1993) and are not presented for the sake of brevity.

accommodate non-smoothness with respect to θ in the moment vector $m(Y(\theta), Z, X, \beta)$, as in, e.g., discrete choice models. The required technical assumptions **A1**-**A7**, along with the proofs of the stated results, which are similar in spirit to and based on Pakes and Pollard (1989), are collected in the Appendix.

Proposition 1 *Let **A1**-**A6**(1) in the Appendix hold. Let S be fixed and $A_N \xrightarrow{P} A$ as $N \to \infty$ where A is positive definite. Then the IPW-II estimator in* (20) *satisfies: $\widehat{\theta}_N^{LM}(A) \xrightarrow{P} \theta^0$.*

Proposition 2 *Let Assumptions **A1**-**A7** in the Appendix hold. Let S be fixed and $A_N \xrightarrow{P} A$ as $N \to \infty$ where A is symmetric and positive definite. Let $\frac{\partial}{\partial \theta'}\beta(\theta^0)$ be full column rank. Then the IPW-II estimator in* (20) *satisfies: $\sqrt{N}(\widehat{\theta}_N^{LM}(A) - \theta^0) \xrightarrow{d} N(0, \Sigma(A))$ where:*

$$\Sigma(A) := \left[\frac{\partial\beta(\theta^0)'}{\partial\theta} G_0' A G_0 \frac{\partial\beta(\theta^0)}{\partial\theta'}\right]^{-1} \frac{\partial\beta(\theta^0)'}{\partial\theta} G_0' A H_0(S) A G_0 \frac{\partial\beta(\theta^0)}{\partial\theta'} \left[\frac{\partial\beta(\theta^0)'}{\partial\theta} G_0' A G_0 \frac{\partial\beta(\theta^0)}{\partial\theta'}\right]^{-1},$$

$$H_0(S) := W_0(S) - C_0\Omega_{22}^{-1}C_0' \equiv \left(1 + \frac{1}{S}\right)[I_0 - K_0] - C_0\Omega_{22}^{-1}C_0'$$

Remarks:

(1) The optimal A is $A^* = H_0^{-1}(S)$. Hence, the optimal asymptotic variance given the auxiliary model is: $\Sigma(A^*) = \left[\frac{\partial\beta(\theta^0)'}{\partial\theta} G_0' H_0^{-1}(S) G_0 \frac{\partial\beta(\theta^0)}{\partial\theta'}\right]^{-1}$. The missing X and the estimation of the nuisance parameters γ to model this missingness make this optimal asymptotic variance different from that given in Proposition 4 of Gourieroux et al. (1993). Without the former, the term $W_0(S)$ in $H_0(S)$ would reduce to the standard definitions given in Gourieroux et al. (1993); i.e., $\xi_{i,S}$, defining the asymptotic expansion of $\sqrt{N}(\widehat{\beta}_N - \widehat{\beta}_{N,S}(\theta^0))$, and hence, $\sqrt{N}(\widehat{\theta}_N^{LM}(A) - \theta^0)$, would reduce to $m(Y_i, Z_i, X_i, \beta^0) - \frac{1}{S}\sum_{s=1}^{S} m(Y_{is}(\theta^0), Z_i, X_i, \beta^0)$. Without the latter, $C_0\Omega_{22}^{-1}C_0'$ would not appear. Under correct specification of the structural model, consistent estimation of $\Sigma(A^*)$ presents no novel theoretical issues: once estimates of γ^0 and $p_0(W_i)$ are obtained, consistent estimates of $H_0(S), G_0$ can be calculated, then consistent estimation of $\Sigma(A^*)$ follows in a similar manner to standard II. However, we note that in contrast to standard II some quantities of $\Sigma(A^*)$ must be estimated using both the observed and simulated data; see, in particular, Ω_{12} and $\Omega_{12}(\theta^0)$ in the definition of C_0. The need to use both simulated and observed data to estimate $\Sigma(A^*)$ is the price we pay for using a (slightly) misspecified simulator; the reader is referred to Dridi et al. (2007) for a thorough discussion on the use of misspecified simulators in II and the resulting theoretical impacts.

(2) The matrix $H_0(S)$ can be written in the equivalent form

$$H_0(S) := E\left[\left(\xi_{i,S} - \Pi[\xi_{i,S}|l_{i,\gamma}(\gamma^0)]\right)\left(\xi_{i,S} - \Pi[\xi_{i,S}|l_{i,\gamma}(\gamma^0)]\right)'\right],$$

$$\xi_{i,S} := \frac{D_i}{p_0(W_i)}\left[m(Y_i, Z_i, X_i, \beta^0) - \frac{1}{S}\sum_{s=1}^{S} m(Y_{is}(\theta^0), Z_i, X_i, \beta^0)\right],$$

where $l_{i,\gamma}(\gamma)$ is the score of the missingness likelihood, and $\Pi[\xi_{i,S}|l_{i,\gamma}(\gamma^0)]$, stands for the affine regression of $\xi_{i,S}$ on $l_{i,\gamma}(\gamma^0)$. Using this formula the optimal asymptotic variance of the IPW-II estimator can be stated as

$$\left[\frac{\partial\beta(\theta^0)'}{\partial\theta} G_0' \left\{E\left[\left(\xi_{i,S} - \Pi[\xi_{i,S}|l_{i,\gamma}(\gamma^0)]\right)\left(\xi_{i,S} - \Pi[\xi_{i,S}|l_{i,\gamma}(\gamma^0)]\right)'\right]\right\}^{-1} G_0 \frac{\partial\beta(\theta^0)}{\partial\theta'}\right]^{-1}.$$

The above is similar to existing formulas describing the asymptotic variance of estimators in the presence of missing data, see, e.g., Wooldridge (2007).

(3) The IPW-II estimator is based on inverse probability weighting the so called "complete cases", i.e., sample units with no missing variables, to correct for the endogenous missingness/selection. This makes it widely applicable to scenarios where the pattern of missingness is more complex [see Little and Rubin (2002)]. For example, let $X = (X_1', X_2')'$ and suppose we observe $(Y', Z')'$ for some sample units, $(Y', Z', X_1')'$ for some and $(Y', Z', X')'$ for the rest. This is a scenario of monotonic pattern in missingness. If there is another subset of the sample units where we observe $(Y', Z', X_2')'$, then this is a scenario of non-monotonic pattern in missingness. The above algorithm can be directly applied under both scenarios since it works with the "complete cases" only, i.e, sample units for which we observe $(Y', Z', X')'$. However, the estimator will not be semiparameterically efficient in the sense of Robins et al. (1994) and Robins and Rotnitzky (1995). Since the driving force behind the potential loss in efficiency related to Remarks (2) and (3) above are well understood now, we abstract from such efficiency considerations to keep this paper short.

(4) Under the tenants of Propositions 1 and 2, the Wald-type IPW-II estimator $\widehat{\theta}_N^W(G_0 A G_0')$, discussed previously, satisfies $\|\widehat{\theta}_N^{LM}(A) - \widehat{\theta}_N^W(G_0 A G_0')\| = o_P(N^{-1/2})$. This result can be proven using results similar to Corollary 3.2 and Theorem 3.4 of Pakes and Pollard (1989). Since this asymptotic equivalence follows standard calculations, in the name of space we refrain from formally stating the result.

3.5 Smoothed Implementation: IPW-GII Estimator

Implementation of the IPW-II estimator when $M_{N,S}(\beta, \gamma, \theta)$ is non-smooth in θ can be computationally burdensome. Following Keane and Smith (2005) and Bruins et al. (2016), we consider an alternative estimator that simplifies estimation via smoothing. The smoothed estimator is obtained in the same manner as $\widehat{\theta}_N^{LM}(A)$, except that $Y_{is}^D(\theta)$ in the original algorithm is replaced by a transformation $Y_{is}^D(\theta, h_N)$ that is smooth (continuously differentiable) in θ for some user-specified sequence of constants $h_N > 0$, where

$$\lim_{h_N \to 0} Y_{is}^D(\theta, h_N) = Y_{is}^D(\theta) \text{ for all } s = 1, \ldots, S \text{ and } i = 1, \ldots, N. \tag{21}$$

The term h_N controls the smoothness of the transformation – larger (smaller) h_N leads to a more (less) smooth transformation but increases (decreases) estimation bias – and needs to be specified by the user taking into consideration the sample size N and the simulation size S.

Such transformations are widely used in simulation-based estimation of discrete choice models to avoid computational difficulties arising from the non-differentiability of the concerned estimating equations with respect to θ (see Train, 2009). To our knowledge, Keane and Smith (2005) were first to propose its use in the context of II. They named the ensuing II procedure Generalized Indirect Inference (GII). Bruins et al. (2016) present a thorough theoretical exposition of GII.

We formally define the GII (smoothed) estimator $\widetilde{\theta}_N^h(A)$ as a solution of:

$$\left\| M_{N,S}^h\left(\widehat{\beta}_N, \widehat{\gamma}_N, \widetilde{\theta}_N^h(A_N)\right)\right\|_{A_N} \;\leq\; o_P\left(N^{-1/2}\right) + \inf_{\theta \in \Theta} \left\| M_{N,S}^h\left(\widehat{\beta}_N, \widehat{\gamma}_N, \theta\right)\right\|_{A_N}, \tag{22}$$

where $M_{N,S}^h(\beta, \gamma, \theta) := \frac{1}{NS} \sum_{i=1}^N \frac{D_i}{p(W_i;\gamma)} \sum_{s=1}^S m(Y_{is}^D(\theta, h_N), Z_i, X_i, \beta,)$ and refer to $\widetilde{\theta}_N^h(A)$ as the IPW-GII estimator of θ^0.

The proposed smoothing approach in Keane and Smith (2005) and Bruins et al. (2016) is more sophisticated than (22) and involves choosing the appropriate smoothing parameter h_N in two steps, which is not fully reflected in the definition (22). In our Monte Carlo experiment involving estimation of structural parameters in a multinomial probit model, however, a naive one-step choice of h_N for the IPW-GII estimator provides significant improvements over the IPW-II estimator. In particular, not only does it reduce the computational cost substantially but it also improves the asymptotic normality approximation for the distribution of the II estimator.[12]

Asymptotic equivalence between $\widetilde{\theta}_N^h(A)$ and $\widehat{\theta}_N^{LM}(A)$ is ensured by letting $h_N \to 0$ at a controlled rate ($\sqrt{N}h_N = o(1)$) and under additional, but standard, technical conditions on the quantities depending on h_N. We collect these conditions as Assumption **A8** in the Appendix.

Proposition 3 *Under **Assumptions A1-A8** in the Appendix, for some sequence of non-negative real numbers h_N satisfying $\sqrt{N}h_N = o(1)$,*

$$\sqrt{N}\left(\widehat{\theta}_N^{LM}(A) - \widetilde{\theta}_N^h(A)\right) = o_P(1).$$

4 Illustrative Example: Multinomial Probit Model

Herein, we consider a multinomial probit model similar to Section 9 in Gourieroux et al. (1993). Our choice of the auxiliary model is the linear probability model, which has similarities with the auxiliary models in Li (2010) and Bruins et al. (2016). In particular, Bruins et al. (2016) use this auxiliary model to estimate the parameters of a multinomial probit model. Section 4.1 specifies the auxiliary model for II and establishes the identification conditions **A2** and **A3** (in the appendix) without explicit consideration of the missing variables. However, missing variables under MAR-X can be accommodated by simply replacing the moment vector for the auxiliary model by its inverse probability weighted version. The satisfaction of **A2** and **A3** ensure the adequacy of the auxiliary model for use in II. Section 4.2 presents a simulation study demonstrating the effectiveness in finite samples of the IPW-II and IPW-GII estimators in this model when the exogenous variable X is missing endogenously following MAR-X in (1).

Additional numerical results for dynamic discrete choice models are also obtained using this same class of auxiliary models presented in Section 4. For brevity and simplicity, we present these additional results in Appendix B and focus herein on the multinomial probit model.

4.1 Indirect Inference: Multinomial Probit Model

Consider a $(J+1)$-alternative multinomial probit model with the alternative 0 as the baseline:

$$\begin{aligned} Y_j &= 1\left(U_j > \max(0, U_k : k = 1, \ldots, J \text{ and } k \neq j)\right), \text{ for } j = 1, ..., J \\ U_j &= Z_j'\alpha + X'\lambda_j + e_j, \end{aligned} \tag{23}$$

and $(e_1, \ldots, e_J)' = \Omega^{1/2}(\varepsilon_1, \ldots, \varepsilon_J)'$ with $\Omega^{1/2}$ lower triangular such that $\Omega^{1/2}\Omega^{1/2'} = \Omega$.

[12] With minor modifications to the assumptions and the theoretical results presented in this paper one can also accommodate the two-step procedure for the choice of h_N, if needed, following the results in Bruins et al (2015).

Let $(\varepsilon_1, \ldots, \varepsilon_J)' \sim N(0, I_J)$ be independent of $Z = (Z_1', \ldots, Z_J')'$, i.e., say the alternative dependent variables, and X, i.e., say the purely individual specific variables.[13] The structural parameters are $\theta = (\alpha', \lambda_1', \ldots, \lambda_J', \omega')'$, where ω are the unique unrestricted elements of Ω.

Our implementation of II in this multinomial probit model follows the same steps described in Section 3.1-3.3. One possible choice for $m(.)$, which we follow in the Monte-Carlo experiment in Section 4.2, is to take:

$$m(R, Z, X, \beta) = \begin{bmatrix} \begin{pmatrix} \zeta(R_1 - \zeta'\beta_1) \\ \vdots \\ \zeta(R_J - \zeta'\beta_J) \end{pmatrix} \\ \text{vech}\left[\begin{pmatrix} R_1 - \zeta'\beta_1 \\ \vdots \\ R_J - \zeta'\beta_J \end{pmatrix} \begin{pmatrix} R_1 - \zeta'\beta_1 \\ \vdots \\ R_J - \zeta'\beta_J \end{pmatrix} - \begin{pmatrix} \beta_{11} & \ldots & \beta_{1J} \\ \vdots & \vdots & \vdots \\ \beta_{1J} & \ldots & \beta_{JJ} \end{pmatrix} \right] \end{bmatrix} \tag{24}$$

where R (stands for response) is either Y or $Y(\theta)$, as appropriate. $R_j = 1(R = j)$ for $j = 1, \ldots, J$ and $\beta = (\beta_1', \ldots, \beta_J', \beta_{11}, \ldots, \beta_{1J}, \beta_{22}, \ldots, \beta_{2J}, \ldots, \beta_{JJ})'$. ζ is some vector valued function of Z and X; for example, $\zeta = (1, Z', X')'$.

This choice of $m(.)$ leads to equation-by-equation ordinary least squares computations in a seemingly unrelated regression (SUR) model with J response variables $1(Y = j)$ or $1(Y(\theta) = j)$ for $j = 1, \ldots, J$; *same set* of regressors ζ for all regressions; and regression errors with unknown variance-covariance matrix. In particular, $m(.)$ represents the vector function specifying the first order conditions for the SUR model regression coefficients, augmented by the estimating equations for the unique elements in the variance-covariance matrix of the SUR regression errors.

Lemma 4 demonstrates that standard least squares identification conditions are sufficient for the key identification conditions **A2** and **A3** to hold in II based on the choice of $m(.)$ in (24). The proof is trivial and hence omitted.

Lemma 4 *Define $Y_j := 1(Y = j)$ and $Y_j(\theta) := 1(Y(\theta) = j)$. Then Assumption **A2** in the Appendix holds if $E[\zeta\zeta']$ is non-singular, while Assumption **A3** in the Appendix holds under the additional orthogonality restriction $E[\zeta(Y_j(\theta) - Y_j(\theta^0)] = 0$ or, equivalently, $E[\zeta(Y_j(\theta) - Y_j)] = 0$ for $j = 1, \ldots, J$ if and only if $\theta = \theta^0$.*

Remarks:

(1) The lemma also applies to other discrete response models as long as the non-singularity and orthogonality conditions hold, such as those considered in Appendix B. This does not contradict the well-known results that, typically such orthogonality (or even mean independence) conditions are not sufficient for non-parametric identification of the structural parameters in discrete response models. While apparently no other distributional assumption has been made in its statement, the lemma is highly parametric and could not possibly be used without knowing the distribution of $Y_j(\theta)$ conditional on Z, X.

(2) Section 4.2 takes $\zeta = (1, Z', X')'$ and, therefore, according to Lemma 4 it implicitly requires for identification of θ^0 the following high level orthogonality conditions:

[13] Normality of ε rules out ties in U_j's almost surely in Z and X. Also assume that the usual restrictions for identification, such as standardizing α, λ_j's and Ω with respect to the $(1,1)$-th element of Ω, and/or any other context specific restrictions are imposed. We abstract from all such issues that are peripheral to the message of our paper.

(a) $P(Y_j(\theta)=1) = P(Y_j(\theta^0)=1)$ for all $j=1,\ldots,J$ if and only if $\theta=\theta^0$.

(b) $E[Z(P(Y_j(\theta)=1|Z,X) - P(Y_j(\theta^0)=1|Z,X))] = 0$ for all $j=1,\ldots,J$ if and only if $\theta=\theta^0$.

(c) $E[X(P(Y_j(\theta)=1|Z,X) - P(Y_j(\theta^0)=1|Z,X))] = 0$ for all $j=1,\ldots,J$ if and only if $\theta=\theta^0$.

(3) A "richer" ζ, for example, that also includes quadratic terms in Z and X, would impose additional such orthogonality conditions and thereby would lead to higher precision of the II estimates. See Bruins et al. (2016) for a careful demonstration of this fact.

(4) The result directly applies to our framework of endogenously missing exogenous variables X by replacing $m(R,Z,X,\beta)$ in (24) by $\frac{D}{p(W;\gamma^0)}m(R,Z,X,\beta)$ and appealing to MAR-X if $R=Y$, or by following similar arguments to those in equation (10) if $R=Y(\theta)$.

Finally, Lemma 4 can also be used to identify the pseudo-true θ (call it θ^*) estimated by II when the exogenous variables X are missing endogenously following MAR-X in (1) and the missingness is simply ignored. Hereafter, we will refer to an II procedure that simply ignores the missingness as standard II.

Consider the following toy example where, for simplicity of demonstration, we take $J=1$, ignore Z, and make specific and convenient distributional assumptions that are covered by our maintained assumptions.

Toy Example: Let the structural model and the missingness mechanism be characterized by:

$$Y = 1(X\lambda^0 + \varepsilon \geq 0) \quad \text{and} \quad D = 1(Y\gamma^0 + v \geq 0)$$

where the scalar random variable X, the structural error ε and the missingness error v are assumed to be independent. Let $\theta^0=\lambda^0$. Following (24), define $m(R,X,\beta) = X(R - X\beta)$ for $R=Y$ or $R=Y(\theta)$. We ignore the second set of moment restrictions in (24) for simplicity.

Therefore, standard II defines $\tilde{\beta}^0$ and $\tilde{\beta}^0(\theta)$ as follows:

$$\tilde{\beta}^0 \text{ solves } E[DX(Y - X\beta)] = 0, \text{ and } \tilde{\beta}^0(\theta) \text{ solves } E[DX(Y(\theta) - X\beta)] = 0.$$

These are essentially the population version of the first two steps of standard II. The final step obtains θ^* by the matching exercise $\tilde{\beta}^0 = \tilde{\beta}^0(\theta^*)$, which by Lemma 4 holds if and only if $E[DXY(\theta^*)] = E[DXY]$. Letting F_T denote the distribution function of any variable T, we know:

$$\begin{aligned} E[DXY(\theta^*)] &= E\left[((1-F_v(-\gamma^0))(1-F_\epsilon(-X\theta^0)) + (1-F_v(0))F_\epsilon(-X\theta^0))(1-F_\epsilon(-X\theta^*))X\right], \\ E[DXY] &= E\left[(1-F_v(-\gamma^0))(1-F_\epsilon(-X\theta^0))X\right]. \end{aligned}$$

The above equalities follow from using MAR-X in (1), the conditional (on X) independence between Y and $Y(\theta)$, and the fact that $F_\varepsilon = F_{\tilde{\varepsilon}}$. For simplicity, assume the specific and convenient distributions: $\varepsilon \sim N(0,1), v \sim N(0,1)$ and $X \sim \text{Bernoulli}(q)$. Denote the distribution function of $N(0,1)$ by $\Phi(.)$ and its inverse by $\Phi^{-1}(.)$. Equating $E[DXY(\theta^*)] = E[DXY]$, standard II yields

$$\text{(pseudo-true value)} \quad \theta^* = \Phi^{-1}\left(\frac{\Phi(\gamma^0)\Phi(\theta^0)}{\Phi(\gamma^0)\Phi(\theta^0) + \Phi(0)(1-\Phi(\theta^0))}\right) \neq \theta^0 \quad \text{(true value)},$$

unless $\gamma^0 = 0$, i.e., unless the missingness is exogenous. Hence, it is the endogeneity of the missingness that causes the problem of identification with standard II. Our proposed IPW-II estimator solves this problem.

4.2 Simulation Study: Three Alternative ($J = 2$) Probit Model

The simulation design considered here is similar to Model 4 in Keane and Smith (2005) and Bruins et al. (2016). In particular, we consider the multinomial probit model in (23) with $J = 2$ for simplicity. For each $i = 1, \ldots, N$, we generate the exogenous regressors as: $Z_{ji} \overset{\text{i.i.d.}}{\sim} \chi^2_1 - 1$ for $j = 1, 2$ and $X_i \overset{\text{i.i.d.}}{\sim} N(1, 2)$ independent of each other. Normalizing all the parameters in the model by the (1,1)-th element of Ω, i.e., equivalently, by fixing $\omega_{11} = 1$ (not to be estimated), we take $\theta^0 = (\alpha^0 = 1, \lambda^0_1 = 1, \lambda^0_2 = 2, \omega^0_{12} = .5, \omega^0_{22} = 1)'$. We generate the structural errors $\varepsilon_i \overset{\text{i.i.d.}}{\sim} N(0, I_2)$ and $e_i = \Omega^{0^{1/2}}\varepsilon_i$ independent of the regressors Z_{1i}, Z_{2i}, X_i and, finally, we generate the outcome Y_i following (23) for each $i = 1, \ldots, N$.

We consider the following missingness mechanism that determines the observability of X. Generate

$$D_i = 1(\gamma^0_1 \times 1(Y_i = 1) + \gamma^0_2 \times 1(Y_i = 2) + \gamma^0_3 \times Z_{2i} \geq v_i)$$

for each $i = 1, \ldots, N$ with $v_i \overset{\text{i.i.d.}}{\sim} N(0, 1)$ independent of the structural errors e_i and the exogenous variables $Z_i = (Z_{1i}, Z_{2i})'$ and X_i. Hence, MAR-X in (1) holds. Take $\gamma^0_1 = -.5, \gamma^0_2 = .5$ and $\gamma^0_3 = 1$. This leads to roughly 50% of sample units with missing X.

We consider the auxiliary model and parameters as defined by the moments $m(.)$ in (24) with $\zeta = (1, Z_1, Z_2, X)'$. Four different LM-type II estimators are considered: the standard II estimator, an infeasible II estimator, the IPW-II and IPW-GII estimators introduced in Section 3. The standard II estimator works with the complete case data $\{D_iY_i, D_iZ_i, D_iX_i\}^N_{i=1}$, i.e., sample units without any missing variables. Standard II ignores the endogenous missingness of X and thus can be biased, gauging the magnitude and consequences of this bias is the first purpose of the simulation study. The infeasible II estimator works with the infeasible full data set $\{Y_i, Z_i, X_i\}^N_{i=1}$, which is only available because we have generated the data, and is not available in practice. The infeasible II is the II estimator that one would use if there were no missing data. Its finite-sample behavior provides an infeasible benchmark for the performance of II in this context. The IPW-II and IPW-GII estimators work with the observed data $\{D_i, Y_i, Z_i, D_iX_i\}^N_{i=1}$ but account for the endogeneity in the missingness of X. These estimators are designed to correct the bias of the standard II estimator, and demonstrating this is the second purpose of the simulation study. The third purpose of the study is to add a word of caution by demonstrating that the asymptotic normal approximation for the IPW-II estimator in Proposition 2 may be inadequate even in reasonably large samples. Lastly, the simulation study demonstrates that the IPW-GII estimator, thanks to the smoothing proposed by Keane and Smith (2005) and Bruins et al. (2016), does not suffer from this issue and is computationally much faster to implement than the other IPW-II estimator.

We compute the mean bias (MBIAS), mean absolute bias (ABIAS), standard deviation (STD), interquartile range (IQR) and the coverage of a 95% Wald-confidence interval (COV95) for all the estimators of $(\alpha^0, \lambda^0_1, \lambda^0_2, \omega^0_{12}, \omega^0_{22})$ for sample sizes $N = 200, 500, 1000$ and 5000. We take $S = 10$ for all estimators. The standard II, infeasible II and IPW-II estimators are computed by the *patternsearch* routine in Matlab. On the other hand, the smoothness of the optimization problem for the IPW-GII estimator allows the use of the gradient-based Matlab routine *fminunc*. Following Bruins et al. (2016), the initial value is set at the true parameter value for all four estimation procedures. All four estimators use the (estimator specific) optimal weighting matrix (see Proposition 2), and in effect are continuously updated GMM estimators. All results are based on $10,000$ Monte-Carlo trials.

To abstract from biases due to small sample sizes and instead focus on the bias that arises because the standard II estimator deliberately ignores the endogenous missingness, we only report the results for the standard II estimator based on $N = 5000$ in Table 1.[14]

θ	MBIAS	ABIAS	STD	IQR	COV95
α	0.0331	0.0472	0.0647	0.0727	94.05
λ_1	0.0259	0.0487	0.0648	0.0698	91.73
λ_2	0.4905	0.4905	0.1013	0.5008	1.03
ω_{12}	-0.1297	0.2053	0.2216	0.2568	91.82
ω_{22}	1.2701	1.2707	0.4538	1.3488	17.54

Table 1: Monte-Carlo results for the Multinomial probit ($J = 2$) model. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, (Monte-Carlo) standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the standard II estimator for the different elements of θ when $N = 5000$.

This estimator is badly biased (see MBIAS) and as a consequence, the coverage of the 95% confidence intervals for the unknown parameters can be extremely low. This is most apparent for the parameter estimates of λ_2 and ω_{22}, which are significantly impacted by the missing data mechanism (through the occurrence of Z_{2i} in the definition of D_i) and for which a nominal coverage of 95% leads to actual coverages of approximately 1% and 17.5% receptively.

Table 2 reports the results for the other three estimators. As expected from the results in Section 3, the IPW-II corrects the bias of the standard II estimator. Its bias (MBIAS) decreases considerably as the sample size increases. ABIAS, STD and IQR also display similar pattern with the increase in sample size. The coverage (COV95) is good. Overall, keeping in mind that X is missing for roughly 50% sample units, the finite-sample behavior of the IPW-II estimator does not deviate much from that of the infeasible benchmark provided by the infeasible II estimator, especially when the sample size is not too small.

Similar phenomenon of bias correction is observed for the IPW-GII estimator. The ABIAS, STD and IQR of IPW-GII are also comparable to the IPW-II estimator.[15] Indeed, the IPW-GII estimator serves the dual purpose stated in Section 3. The IPW-GII estimator is much faster than the IPW-II estimator, and more importantly, while the studentized IPW-II estimator is far from from being normally distributed, even for sample size $N = 5000$, no such problem arises for the IPW-GII estimator;[16] Figure 1 gives precise details.

[14]Results for other sample sizes are available from the authors.

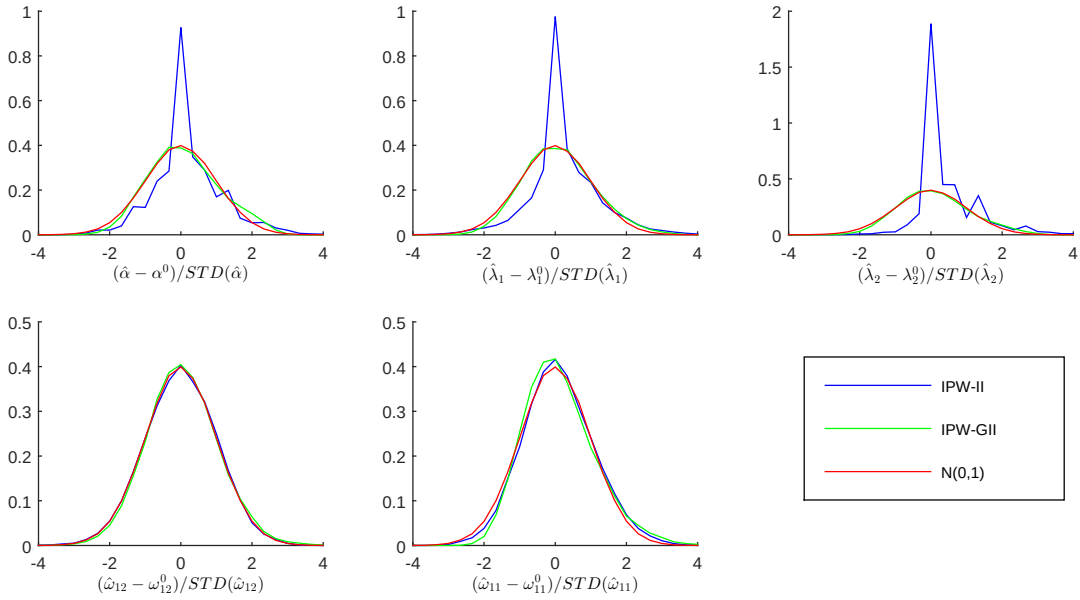
[15]The smoothing parameter h_N is .078, .0571, .0458, .0284 respectively for $N = 200, 500, 1000, 5000$. This is in rough accordance to the requirements of Proposition 3 but with a slight tilt toward zero for the smaller sample sizes $N = 200, 500$ to reduce the bias due to smoothing. While the choice of h_N is important for the performance of GII, given the scope of our paper, we defer discussion on the choice of h_N in practical settings to Bruins et al. (2016) who give a thorough analysis on how h_N can be chosen in practice.

[16]The same issue is also present in the infeasible II estimator. However, for both the infeasible II and IPW-II, the quality of the normal approximation is better if we use a richer auxiliary model by augmenting $\zeta = (1, Z', X')'$ with quadratic terms in Z and X. This removes some wiggliness in the corresponding kernel density plots. These figures are not included for brevity but can be found in the previous version of the paper and are available from the authors.

		N = 200					N = 500				
Estimator	θ	MBIAS	ABIAS	STD	IQR	COV95	MBIAS	ABIAS	STD	IQR	COV95
	α	0.0576	0.1063	0.1540	0.1645	93.36	0.0243	0.0617	0.0938	0.0969	91.84
	λ_1	0.0330	0.1000	0.1463	0.1500	93.45	0.0239	0.0672	0.0984	0.1013	92.82
Infeasible II	λ_2	0.0924	0.1119	0.1653	0.1894	91.94	0.0959	0.3368	0.4767	0.5154	94.86
	ω_{12}	-0.0513	0.2740	0.3399	0.3438	95.54	0.0519	0.0705	0.1022	0.1146	90.69
	ω_{22}	0.0852	0.4624	0.6028	0.6088	95.29	-0.0047	0.1596	0.2003	0.2003	94.64
	α	0.1237	0.2100	0.3758	0.3957	94.81	0.0648	0.1121	0.1794	0.1908	93.16
	λ_1	0.0875	0.1723	0.3133	0.3253	95.55	0.0437	0.1126	0.1803	0.1856	93.49
IPW-II	λ_2	0.2425	0.2683	0.5282	0.5812	95.53	0.1021	0.1201	0.2084	0.2321	93.71
	ω_{12}	-0.0634	0.4215	0.5195	0.5233	94.07	-0.0152	0.3018	0.3690	0.3694	96.23
	ω_{22}	0.3961	0.8948	3.3841	3.4072	99.17	0.1016	0.5535	0.9864	0.9916	98.20
	α	0.3230	0.3253	0.7685	0.7146	92.30	0.0802	0.1829	0.3974	0.3674	95.05
	λ_1	0.3190	0.2753	0.6837	0.6283	91.62	0.0746	0.1498	0.3196	0.3038	94.45
IPW-GII	λ_2	0.6686	0.5356	1.3933	1.1999	91.39	0.1585	0.1861	0.6308	0.3838	94.22
	ω_{12}	0.1954	0.3952	0.7685	0.8270	92.98	0.1071	0.2464	0.4486	0.5040	93.42
	ω_{22}	1.5954	0.7944	3.5950	2.2757	93.56	0.4260	0.4769	1.3016	1.0343	94.45
		N = 1000					N = 5000				
Estimator	θ	MBIAS	ABIAS	STD	IQR	COV95	MBIAS	ABIAS	STD	IQR	COV95
	α	0.0156	0.0463	0.0664	0.0682	94.63	0.0070	0.0222	0.0310	0.0318	91.16
	λ_1	0.0141	0.0493	0.0727	0.0741	92.95	0.0072	0.0268	0.0372	0.0379	93.12
Infeasible II	λ_2	0.0302	0.0427	0.0645	0.0712	91.51	0.0146	0.0225	0.0320	0.0351	91.29
	ω_{12}	-0.0031	0.1162	0.1465	0.1465	95.21	-0.0021	0.0563	0.0709	0.0709	94.83
	ω_{22}	0.0343	0.1928	0.2452	0.2476	94.56	0.0137	0.0901	0.1135	0.1143	94.56
	α	0.0387	0.0866	0.1279	0.1336	94.01	0.0106	0.0354	0.0513	0.0523	93.37
	λ_1	0.0269	0.0809	0.1228	0.1257	91.88	0.0148	0.0421	0.0608	0.0626	92.05
IPW-II	λ_2	0.0632	0.0744	0.1150	0.1312	90.23	0.0248	0.0315	0.0458	0.0522	91.96
	ω_{12}	-0.0086	0.2403	0.2960	0.2962	96.81	-0.0002	0.1155	0.1451	0.1451	94.97
	ω_{22}	0.0684	0.4245	0.5475	0.5517	95.72	0.0250	0.1987	0.2515	0.2527	94.72
	α	0.0071	0.1037	0.1916	0.2073	94.28	0.0128	0.0800	0.1212	0.1617	94.62
	λ_1	0.0040	0.0829	0.1334	0.1653	94.86	0.0099	0.0690	0.1019	0.1386	94.55
IPW-GII	λ_2	0.0161	0.063	0.2137	0.1279	93.53	0.0236	0.1307	0.1967	0.2638	94.35
	ω_{12}	0.0618	0.1302	0.2788	0.2682	93.06	0.0063	0.0873	0.1332	0.1747	94.94
	ω_{22}	0.1367	0.2015	0.5596	0.4254	94.01	0.0386	0.1954	0.3036	0.3973	94.86

Table 2: Monte-Carlo results for the Multinomial probit ($J = 2$) model. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the concerned estimator for the different elements of the parameter vector θ. STD is not based on the asymptotic variance formula in Proposition 2 but on the Monte-Carlo. Results are obtained by 10,000 Monte-Carlo trials.

Figure 1: Kernel density plots for the studentized IPW-II and IPW-GII estimators of the different elements of the parameter vector θ when sample size $N = 5000$. Results are reported based on 10,000 Monte Carlo trials. It is clear that IPW-II can be far away from the standard normal approximation (red line) while IPW-GII does not suffer from this problem.



5 Conclusion

In this paper we have demonstrated the problems with identification and consistent estimation of the structural parameters by II when the exogenous variables can be endogenously missing following the MAR-X assumption, which can arise in empirical work for reasons such as survey non-response, survey revisions, cost-effective survey design, etc. Our proposed solution can be implemented as either the IPW-II or the IPW-GII estimator, with the smoothed IPW-GII approach being particularly useful in non-smooth problems. This novel estimation method corrects for the sample selection bias in the estimation of the auxiliary parameters with the observed data and the simulated data using the method of inverse probability weighting. The desirable performance of the proposed II approach was demonstrated theoretically and via simulations. The extremely poor performance of standard II estimators that simply discard sample units with missingness was also demonstrated via simulations. We conclude by noting that the selection due to the missing data is handled by our proposed method in one step that only involves the estimation of the missingness (conditional) probabilities using a binary choice model, such as logit or probit, and hence the proposed method retains the computational attractiveness of II procedures.

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A Appendix: Technical assumptions and proofs

A.1 Technical Assumptions

The following notations are used. For a $d \times d$ matrix A and a $c \times d$ matrix B, define $\|B\|_A := \sqrt{\text{Trace}(BAB')}$ and $\|B\| := \|B\|_{A=I_d}$. Define $\mathcal{N}_\delta(\theta^0) \subset \Theta$, $\mathcal{N}_\delta(\beta^0) \subset \mathcal{B}$ and $\mathcal{N}_\delta(\gamma^0) \subset \Gamma$ as some generic open neighborhoods of radius δ for θ^0, β^0 and γ^0 respectively. Finally, define

$$M(\beta,\gamma) := E\left[\frac{D}{p(W;\gamma)} m(Y,Z,X,\beta)\right] \text{ and } M(\beta,\gamma,\theta) := E\left[\frac{D}{p(W;\gamma)} m(Y(\theta),Z,X,\beta)\right].$$

By the definition of β^0 in (4) and the definition of the binding function in (12)

$$M(\beta^0,\gamma^0) = M(\beta^0,\gamma^0,\theta^0) = 0. \tag{25}$$

Assumption A1:

(a) Structural Model in (2): ε has a known distribution $F_\varepsilon = F_\varepsilon^0$ and is independent of Z and X whose unknown distribution is $F_{(Z,X)} = F_{(Z,X)}^0$.

(b) Strict overlap: For MAR in (1), $p_0(W) := P(D=1|W) \in [p,1)$ for a constant $p > 0$.

(c) Observed sample: $\{W_i, D_i, D_i X_i\}_{i=1}^N$ are i.i.d. copies of W, D, and DX.

Assumption A2: β^0 is the unique solution to equation (4).
Assumption A3: For all $\theta \in \Theta$, the binding function $\beta(\theta)$ defined in equation (12) satisfies $\beta^0 = \beta(\theta)$ if and only if $\theta = \theta^0$.
Assumption A4 : There exists a unique $\gamma^0 \in \Gamma$ and a function $p(w,\gamma) : \text{Support}(W) \times \Gamma \mapsto (0,1)$ such that $p_0(w) = p(w;\gamma^0)$ for all $w \in \text{Support}(W)$. $\Gamma \subset \mathbb{R}^{d_\gamma}$ is compact and d_γ is finite.
Assumption A5:

(a) $\Theta \subset \mathbb{R}^{d_\theta}$ and $\mathcal{B} \subset \mathbb{R}^{d_\beta}$ are compact with $\theta^0 \in \text{interior}(\Theta)$ and $\beta^0 \in \text{interior}(\mathcal{B})$.

(b) For $l = (l_1, l_2, l_3)$ where $l_1 \in \text{Support}(Y \text{ or } Y(\theta))$ (as appropriate) and $(l_2, l_3) \in \text{Support}(Z, X)$: $m(l,\beta)$ is continuous in β for all l, and $\|m(l,\beta)\|^2 \le g(l)$ for all l and $E[g(l)] < \infty$.

(c) For $\delta > 0$: $\sup_{\theta\in\Theta, \beta\in\mathcal{N}_\delta(\beta^0), \gamma\in\mathcal{N}_\delta(\gamma^0)} \dfrac{\|M_{N,S}(\beta,\gamma,\theta) - M(\beta,\gamma,\theta)\|}{1 + \|M_{N,S}(\beta,\gamma,\theta)\| + \|M(\beta,\gamma,\theta)\|} = o_P(1)$.

Assumption A6:

(a) $p(w;\gamma)$ is continuous in $\gamma \in \Gamma$ for all $w \in \text{Support}(W)$.

(b) For some $\delta > 0$: $p(w;\gamma)$ is twice continuously differentiable in $\gamma \in \mathcal{N}_\delta(\gamma^0)$ for all $w \in \text{Support}(W)$, and the derivatives $p_\gamma(w;\gamma) := \frac{\partial}{\partial\gamma'} p(w;\gamma)$ and $p_{\gamma\gamma}(w;\gamma) := \frac{\partial}{\partial\gamma'} p'_\gamma(w;\gamma)$ satisfy: $\sup_{\gamma\in\mathcal{N}_\delta(\gamma^0)} \|p_\gamma(w;\gamma)\|^2 + \sup_{\gamma\in\mathcal{N}(\gamma^0)} \|p_{\gamma\gamma}(w;\gamma)\| < b(w)$ for all $w \in \text{Support}(W)$ where $b(w) \ge 0$ and $E[b(w)] < \infty$.

(c) The score $l_\gamma(D,W;\gamma) := (D - p(W;\gamma))p'_\gamma(W;\gamma)/[p(W;\gamma)(1-p(W;\gamma))]$ is such that $\Omega_{22} = E\left[l_\gamma(D,W;\gamma^0) l'_\gamma(D,W;\gamma^0)\right]$ is nonsingular.

Assumption A7:

(a) For each $l = (l_1, l_2, l_3)$ where $l_1 \in$ Support(Y or $Y(\theta)$) (as appropriate) and $(l_2, l_3) \in$ Support(Z, X), $m(l, \beta)$ is continuously differentiable in $\beta \in \mathcal{N}_\delta(\beta^0)$ for some $\delta > 0$. Allow for changing the order of differentiation and integration, i.e., let $E\left[\sup_{\beta \in \mathcal{N}_\delta(\beta^0)} \|\partial m(l,\beta)/\partial \beta'\|\right] < \infty$.

(b) $G_0 := E\left[\frac{\partial}{\partial \beta'} \frac{D}{p_0(W)} m(Y, Z, X, \beta^0)\right] \equiv E\left[\frac{\partial}{\partial \beta'} \frac{D}{p_0(W)} m(Y(\theta^0), Z, X, \beta^0)\right]$ is nonsingular.

(c) $\sqrt{N}\bar{\xi}_{N,S} \xrightarrow{d} N(0, W_0(S))$ where $W_0(S) = \left(1 + \frac{1}{S}\right)(I_0 - K_0)$, $\bar{\xi}_{N,S} := \sum_{i=1}^N \xi_{i,S}/N$,

$$\xi_{i,S} := \frac{D}{p(W;\gamma^0)} m(Y, Z, X, \beta^0) - \frac{1}{S}\sum_{s=1}^{S} \frac{D}{p(W;\gamma^0)} m(Y(\theta^0), Z, X, \beta^0).$$

(d) $(\partial/\partial \theta')M(\beta^0, \gamma^0, \theta)$ is continuous for $\theta \in \mathcal{N}_\delta(\theta^0)$ and has rank d_θ at $\theta = \theta^0$.

(e) For every positive sequences $\{\delta_N\}$ and $\delta_N = o(1)$

$$\sup_{\theta \in \mathcal{N}_{\delta_N}(\theta^0), \beta \in \mathcal{N}_{\delta_N}(\beta^0), \gamma \in \mathcal{N}_{\delta_N}(\gamma^0)} \frac{\sqrt{N}\|M_{N,S}(\beta,\gamma,\theta) - M(\beta,\gamma,\theta) - M_{N,S}(\beta^0,\gamma^0,\theta^0)\|}{1 + \sqrt{N}\|M_{N,S}(\beta,\gamma,\theta)\| + \sqrt{N}\|M(\beta,\gamma,\theta)\|} = o_P(1).$$

To establish the asymptotic properties of the GII estimator, additionally define for each h:

$$M^h(\beta,\gamma,\theta) := E\left[\frac{D}{p(W;\gamma)} m(Y(\theta,h), Z, X, \beta)\right].$$

As before like (25) and further using (21),

$$M(\beta^0,\gamma^0) = M(\beta^0,\gamma^0,\theta^0) = M^{h=0}(\beta^0,\gamma^0,\theta^0) = 0. \tag{26}$$

The following assumptions on $M_N^h(\beta,\gamma,\theta)$ $M^h(\beta,\gamma,\theta)$ and $M(\beta,\gamma,\theta)$ are additionally maintained for the asymptotic equivalence of the GII and II estimators.

Assumption A8: For some $\delta > 0$ and a finite $b > 0$, let the following hold for $M_{N,S}^h(.)$ and its limit counterpart $M^h(.)$:[17]

(a) $\sup_{\theta \in \Theta, \beta \in \mathcal{N}_\delta(\beta^0), \gamma \in \mathcal{N}_\delta(\gamma^0)} \|M^h(\beta,\gamma,\theta) - M(\beta,\gamma,\theta)\| \le b \times h$ for $h \in [0, \delta)$.

(b) $\sup_{h \in [0,\delta)} \sup_{\theta \in \Theta, \beta \in \mathcal{N}_\delta(\beta^0), \gamma \in \mathcal{N}_\delta(\gamma^0)} \frac{\|M_N^h(\beta,\gamma,\theta) - M^h(\beta,\gamma,\theta)\|}{1 + \|M_N^h(\beta,\gamma,\theta)\| + \|M^h(\beta,\gamma,\theta)\|} = o_P(1).$

(c) (i) $\sup_{h \in (0,\delta)} \sup_{\theta \in \mathcal{N}_\delta(\theta^0), \beta \in \mathcal{N}_\delta(\beta^0), \gamma \in \mathcal{N}_\delta(\gamma^0)} \left\| \frac{\partial}{\partial(\beta',\gamma')} \left(M_N^h(\beta,\gamma,\theta) - M^h(\beta,\gamma,\theta)\right)\right\| = o_P(1).$

(c) (ii) $\sup_{h \in (0,\delta)} \sup_{\theta \in \mathcal{N}_\delta(\theta^0), \beta \in \mathcal{N}_\delta(\beta^0), \gamma \in \mathcal{N}_\delta(\gamma^0)} \left\| \frac{\partial}{\partial \theta'} \left(M_N^h(\beta,\gamma,\theta) - M^h(\beta,\gamma,\theta)\right)\right\| = O_P(N^{-1/2}).$

[17] See Remark 5 for an explanation of these assumptions.

(d) $\frac{\partial}{\partial(\beta',\gamma',\theta')}M^h(\beta,\gamma,\theta)$ is continuous in β,γ,θ, h for $(\beta,\gamma,\theta)\in\mathcal{N}_\delta(\beta^0,\gamma^0,\theta^0)$ and $h\in[0,\delta)$.

(e) $\sup_{h\in(0,\delta)}\dfrac{\sqrt{N}\|M_N^h(\beta^0,\gamma^0,\theta^0)-M^h(\beta^0,\gamma^0,\theta^0)-M_N(\beta^0,\gamma^0,\theta^0)\|}{1+\sqrt{N}\|M_N^h(\beta^0,\gamma^0,\theta^0)\|+\sqrt{N}\|M^h(\beta^0,\gamma^0,\theta^0)\|}=o_P(1)$.

Remark 1: It is well known that by Assumptions **A4** and **A6**, the maximum likelihood estimator $\widehat{\gamma}_N$ that gives $\widehat{p}_0(w)=p(w,\widehat{\gamma}_N)$ for Step 0 of modified II satisfies:

$$\sqrt{N}(\widehat{\gamma}_N-\gamma^0)=\Omega_{22}^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^N l_\gamma(D_i,W_i;\gamma^0)+o_P(1). \tag{27}$$

Also, (1), Assumptions **A1**(b)-(c), **A5**(a)-(b) and **A7**(a)-(b) and (27) give for $\widehat{\beta}_N$ from Step 1:

$$\begin{aligned}\sqrt{N}(\widehat{\beta}_N-\beta^0) &= -G_0^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^N\left\{m_i^*(\gamma^0,\beta^0)-\Omega_{12}\Omega_{22}^{-1}l_{i,\gamma}(D_i,W_i;\gamma^0)\right\}+o_P(1), \qquad (28)\\ m_i^*(\gamma^0,\beta^0) &= \frac{D_i}{p_0(W_i;\gamma^0)}m(Y_i,Z_i,X_i,\beta^0),\ \ \Omega_{12}=E\left[m_i^*(\gamma^0,\beta^0)l_{i,\gamma}(D_i,W_i;\gamma^0)'\right]\end{aligned}$$

See. e.g., Chaudhuri and Min (2012) for (27) and (28). Similar steps and (10) give for $\widehat{\beta}_N(\theta^0)$, defined as

$$\widehat{\beta}_N(\theta^0) \ := \ \arg_{\beta\in\mathcal{B}}\left\{M_{N,S}\left(\beta,\widehat{\gamma}_N,\theta^0\right)=0\right\}, \tag{29}$$

the asymptotically linear representation

$$\begin{aligned}\sqrt{N}(\widehat{\beta}_N(\theta^0)-\beta^0) &= -G_0^{-1}\frac{1}{\sqrt{N}}\sum_{i=1}^N\frac{1}{S}\sum_{s=1}^S\left\{m_{is}^*(\gamma^0,\beta^0;\theta^0)-\Omega_{12}(\theta^0)\Omega_{22}^{-1}l_{i,\gamma}(D_i,W_i;\gamma^0)\right\}+o_P(1), \qquad (30)\\ m_{is}^*(\gamma^0,\beta^0;\theta^0) &= \frac{D_i}{p_0(W_i;\gamma^0)}m(Y_{is}(\theta^0),Z_i,X_i,\beta^0),\ \ \Omega_{12}(\theta^0)=E\left[m_{is}^*(\gamma^0,\beta^0;\theta^0))l_{i,\gamma}(D_i,W_i;\gamma^0)'\right]\end{aligned}$$

Therefore, under Assumption **A7**(c) and for a fixed S, using (27), (28) and (30) jointly give:

$$\begin{aligned}\sqrt{N}(\widehat{\beta}_N-\widehat{\beta}_N(\theta^0)) &= -G_0^{-1}\sqrt{N}\left[\bar{\xi}_{N,S}-C_0(\widehat{\gamma}_N-\gamma^0)\right] \ \xrightarrow{d}\ N\left(0,G_0^{-1}H_0(S)G_0^{-1'}\right) \qquad (31)\\ H_0(S) &= (1+\frac{1}{S})(I_0-K_0)-C_0B_0^{-1}C_0',\ \ C_0=\Omega_{12}-\Omega_{12}(\theta^0)\end{aligned}$$

Remark 2: It is important to point out that the equivalence of the G_0 terms in the expansions (28) and (30), follows since, under the maintained assumptions, for α being any one of β,γ,θ,

$$\frac{\partial M(\gamma^0,\beta^0,\theta^0)}{\partial\alpha'} \ = \ \left\{\frac{\partial}{\partial\alpha'}\frac{1}{S}\sum_{s=1}^S E\left[\frac{D}{p(W;\gamma)}m(Y_s(\theta),Z,X,\beta)\right]\right\}\Big|_{\beta^0,\theta^0,\gamma^0}=\left\{\frac{\partial}{\partial\alpha'}E\left[m(Y_s(\theta),Z,X,\beta)\right]\right\}\Big|_{\beta^0,\theta^0,}$$

which follows by MAR-X and the knowledge that the S simulated samples are iid copies. More generally then, the partial derivatives $\partial M/\partial\theta'$, $\partial M/\partial\beta'$, $\partial M/\partial\gamma'$ (evaluated at the truth) do not depend on S. Likewise, it is important to note the role that S does play in the asymptotic variance of $\bar{\xi}_{N,S}$.

Remark 3: Assumption **A5**(c) is an uniform convergence condition for which the strict overlap

assumption in **A1**(b) plays a crucial role. The same holds for the stochastic equicontinuity condition Assumption **A7**(e) that is similar to condition (iii) in Theorem 3.3 of Pakes and Pollard (1989), but additionally it allows for nuisance parameters close to their true values. Such high-level assumptions need to be verified on a case-by-case basis [see, e.g., Cattaneo (2010) or Chaudhuri and Guilkey (2016)].

Remark 4: Since it is also known how to account for random weighting matrix [see Lemmas 3.4 and 3.5 of Pakes and Pollard (1989)], we abstract from it in all the proofs below and instead directly assume in the concerned propositions that the weighting matrix A_N is possibly based on some preliminary consistent estimators of the concerned parameters such that $A_N \xrightarrow{P} A$ where A is a positive definite matrix. Hence in what follows let $\widehat{\theta}_N := \widehat{\theta}_N^{LM}(A)$.

Remark 5: Assumption **A8** is a high-level condition restricting the choice of kernels (e.g. logistic or normal) used within the smoothing step of the generalized II procedure. Essentially it imposes sufficient smoothness condition on $M_N^h(.)$ and $M^h(.)$ to facilitate simple proofs of the desired asymptotic properties of the GII estimator. The denominators in **A8**(b) and (d) add slightly more generality (similar to those in **A5**(d) and **A7**(e)). The asymmetric treatment with respect to (β, γ) and θ in **A8**(c) (i) and (ii) respectively is due to the fact that we do not formally establish $\sqrt{N}$-consistency of $\widetilde{\theta}_N^h$ prior to demonstrating its asymptotic normality. The stronger condition in (ii) bears resemblance with the assumptions on suitable central limit theorem for Jacobians in the weak identification literature (see Kleibergen (2005)).

A.2 Proofs

Proof of Proposition 1: For notational simplicity, in what follows we drop the S subscript from the definition of $M_{N,S}(\cdot)$ since S is assumed fixed.

The proof proceeds by showing that $\|M(\beta^0, \gamma^0, \widehat{\theta}_N)\| = o_P(1)$. Under Assumptions **A2** and **A3**, this condition is sufficient for $\widehat{\theta}_N \xrightarrow{P} \theta^0$ by virtue of (25), (27), (28) [where the last two give: $\widehat{\gamma}_N \in \mathcal{N}_\delta(\gamma^0)$ and $\widehat{\beta}_N \in \mathcal{N}_\delta(\beta^0)$ respectively with probability approaching 1], and the continuity implied by Assumptions **A1**(b), **A6**(a) and **A5**(b). Note that by the triangle inequality:

$$\begin{aligned}\|M(\beta^0, \gamma^0, \widehat{\theta}_N)\| \leq & \|M(\beta^0, \gamma^0, \widehat{\theta}_N) - M(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\| + \|M(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N) - M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\| \\ & + \|M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\|.\end{aligned}$$

By (27), (28), and the continuity implied by Assumptions **A1**(b), **A6**(a) and **A5**(b), the first term on the right hand side, i.e., $\|M(\beta^0, \gamma^0, \widehat{\theta}_N) - M(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\|$ is $o_P(1)$. (27), (28) and Assumption **A5**(c) imply that the second term $\|M(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N) - M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\|$ is $o_P(1)$. The definition in (20) implies that the third term $\|M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \widehat{\theta}_N)\| \leq \|M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \theta^0)\| = \|M(\widehat{\beta}_N, \widehat{\gamma}_N, \theta^0)\| + o_P(1)$ where the equality follows from (27), (28) and Assumption **A5**(c). Since (27), (28) and, as before, the continuity of $M(\beta, \gamma, \theta^0)$ in β and γ imply that $\|M(\widehat{\beta}_N, \widehat{\gamma}_N, \theta^0)\| = \|M(\beta^0, \gamma^0, \theta^0)\| + o_P(1)$, it follows by (25) that the third term is also $o_P(1)$. ■

Proof of Proposition 2: For notational simplicity, again, in what follows we drop the S subscript from the definition of $M_{N,S}(\cdot)$.

Since $\widehat{\theta}_N \xrightarrow{P} \theta^0$, it follows by (25) and Assumption **A7**(d) that $\|\widehat{\theta}_N - \theta^0\| = O_P\left(\|M(\beta^0, \gamma^0, \widehat{\theta}_N)\|\right)$. Under our maintained assumptions and (27) and (28), it can then be shown that $\|M(\beta^0, \gamma^0, \widehat{\theta}_N)\|$

and hence $\|\widehat{\theta}_N - \theta^0\|$ is $O_P(N^{-1/2})$. Details are available from the authors. Given this, and that our assumptions are essentially same as that in Theorem 3.5 of Pakes and Pollard (1989), the rest of the proof is also similar. Hence we only provide a sketch of the proof below, and highlight the differences that appear only to the end of the proof.

For now let $d_\theta = d_\beta$. Justifying by virtue of (27), (28) and the $\sqrt{N}$-consistency of $\widehat{\theta}_N$, linearize $M_N(\widehat{\zeta}_N, \theta)$ in a $\sqrt{N}$-neighborhood of θ^0 by the function [see, for example, Chen et al. (2003)]:

$$L_N(\theta) := M_N(\beta^0, \gamma^0, \theta^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \beta'}(\widehat{\beta}_N - \beta^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \gamma'}(\widehat{\gamma}_N - \gamma^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \theta'}(\theta - \theta^0).$$

Define $\theta_N^* = \arg\min_\theta \|L_N(\theta)\|$. For the application of Assumption **A7**(e) in the remainder of the proof choose δ_N such that $\widehat{\beta}_N \in \mathcal{N}_{\delta_N}(\beta^0)$, $\widehat{\gamma}_N \in \mathcal{N}_{\delta_N}(\gamma^0)$, and both $\widehat{\theta}_N, \theta_N^* \in \mathcal{N}_{\delta_N}(\theta^0)$. It can now be shown (details available from the authors) by (27), (28), Assumption **A7**(e) and the $\sqrt{N}$-consistency of $\widehat{\theta}_N$ that $\|M_N(\widehat{\beta}_N, \widehat{\gamma}_N, \theta) - L_N(\theta)\| = o_P(N^{-1/2})$ for both $\theta = \widehat{\theta}_N$ and $\theta = \theta_N^*$, and thus, subsequently, by Assumption **A7**(d) that

$$\sqrt{N}(\widehat{\theta}_N - \theta^0) = \sqrt{N}(\theta_N^* - \theta^0) = o_p(1). \tag{32}$$

Now note by (29): $\widehat{\beta}_N(\theta^0)$ satisfies $0 = M_N(\widehat{\beta}_N(\theta^0), \widehat{\gamma}_N, \theta^0)$. Expanding the right hand side gives:

$$0 = M_N(\beta^0, \gamma^0, \theta^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \beta'}(\widehat{\beta}_N(\theta^0) - \beta^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \gamma'}(\widehat{\gamma}_N - \gamma^0) + o_P(N^{-1/2}). \tag{33}$$

On the other hand, since $\theta_N^* = \arg\min_\theta \|L_N(\theta)\|$, it follows that $o_P(N^{-1/2}) = L_N(\theta_N^*)$. Hence by the definition of $L_N(\theta_N^*)$ and using $\sqrt{N}$-consistency of $\widehat{\beta}_N$, $\widehat{\gamma}_N$ and θ_N^* it follows that:

$$o_P(N^{-1/2}) = M_N(\beta^0, \gamma^0, \theta^0) + \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial(\beta', \gamma', \theta')}\left[(\widehat{\beta}_N - \beta^0)', (\widehat{\gamma}_N - \gamma^0)', (\theta_N^* - \theta^0)'\right]'. \tag{34}$$

Therefore, equating (33) and (34) gives:

$$\frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \theta'}\sqrt{N}(\theta_N^* - \theta^0) = -\frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \beta'}\sqrt{N}(\widehat{\beta}_N - \widehat{\beta}_N(\theta^0)) + o_P(1).$$

Until now in this proof we have disregarded the over-identifying nature of the system with respect to θ. However, when $d_\theta < d_\beta$, and $A_N \xrightarrow{P} A$ (positive definite), under Assumption **A7**(d), standard methods modify the above relation as, up to an $o_P(1)$ term:

$$\frac{\partial M'(\beta^0, \gamma^0, \theta^0)}{\partial \theta} A \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \theta'}\sqrt{N}(\theta_N^* - \theta^0) = -\frac{\partial M'(\beta^0, \gamma^0, \theta^0)}{\partial \theta} A \frac{\partial M(\beta^0, \gamma^0, \theta^0)}{\partial \beta'}\sqrt{N}(\widehat{\beta}_N - \widehat{\beta}_N(\theta^0)).$$

From the above expression, and **Remark 3**, we see that θ_N^*, and hence $\widehat{\theta}_N$, depends on S only through the dependence of $\bar{\xi}_{N,S}$ on S.

Differentiating $M(\beta(\theta), \gamma^0, \theta^0)$ with respect to θ at $\theta = \theta^0$ and using Assumption **A7**(d):

$$\frac{\partial}{\partial \theta'} M(\beta(\theta^0), \gamma^0, \theta^0) = \frac{\partial}{\partial \beta'} M(\beta(\theta^0), \gamma^0, \theta^0) \frac{\partial}{\partial \theta'} \beta(\theta^0) = G_0 \frac{\partial}{\partial \theta'} \beta(\theta^0)$$

where the last equality follows by Assumption **A3**, **A4**, **A7**(a), (b) and MAR-X in (1). Combining the above and using (31) and (32) we obtain:

$$\sqrt{N}(\widehat{\theta}_N - \theta^0) = \left[\frac{\partial\beta(\theta^0)'}{\partial\theta}G_0'AG_0\frac{\partial\beta(\theta^0)}{\partial\theta'}\right]^{-1}\frac{\partial\beta(\theta^0)'}{\partial\theta}G_0'A\sqrt{N}\left[\bar{\xi}_{N,S} - C_0(\widehat{\gamma}_N - \gamma^0)\right] + o_P(1)$$

where $C_0 = \Omega_{12} - \Omega_{12}(\theta^0)$ and $\bar{\xi}_{N,S} = \frac{1}{N}\sum_{i=1}^N \xi_{i,S} \equiv \frac{1}{N}\sum_{i=1}^N \left[m_i^*(\gamma^0,\beta^0) - \frac{1}{S}\sum_{s=1}^S m_{is}^*(\gamma^0,\beta^0;\theta^0)\right]$. Again, by similar arguments to those in **Remark 3**, $\sqrt{N}(\widehat{\theta}_N - \theta^0)$ depends on S only through the dependence of $\bar{\xi}_{N,S}$ on S, which under the maintained assumptions yields

$$\sqrt{N}(\widehat{\theta}_N - \theta^0) \to_d N(0, \Sigma(A)).\blacksquare$$

Proof of Proposition 3: For notational simplicity, we will drop the N subscript from h (with the understanding that for any given N, $h > 0$ but $h = o(N^{-1/2})$) and the S subscript from the definition of $M_{N,S}^h(\cdot)$. Also, since the weighting matrix A_N can be handled in the same manner as in **Proposition** 2, we only consider the just-identified case ($d_\theta = d_\beta$) and take $A_N = A = I_{d_\beta}$. The proof now proceeds in two steps, first we demonstrate consistency of $\widetilde{\theta}_N^h$ for θ^0, and we then demonstrate $\|\widetilde{\theta}_N^h - \widehat{\theta}_N\| = o_P(N^{-1/2})$. The entire proof closely follows that of **Propositions** 1 and 2 except that having established consistency we slightly deviate to emphasize the fact that $M_N^h(\beta,\gamma,\theta)$ is indeed differentiable with respect to θ for $h > 0$.

Consistency: Following **Proposition** 1, by continuity of $M(\beta,\gamma,\theta)$ in θ, the result follows if $\|M(\beta^0,\gamma^0,\widetilde{\theta}_N^h)\| = o_P(1)$ as $h \to 0$. This condition will be sufficient for $\widetilde{\theta}_N^h \xrightarrow[h\to 0]{P} \theta^0$ by the same arguments as **Proposition** 1. By the triangle inequality:

$$\begin{aligned}\|M(\beta^0,\gamma^0,\widetilde{\theta}_N^h)\| \leq & \|M(\beta^0,\gamma^0,\widetilde{\theta}_N^h) - M(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\| + \|M(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h) - M^h(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\| \\ & + \|M^h(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h) - M_N^h(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\| + \|M_N^h(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\|. \qquad (35)\end{aligned}$$

As before, by Assumptions **A1**(b), **A6**(a) and **A5**(b), $\|M(\beta^0,\gamma^0,\widetilde{\theta}_N^h) - M(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\|$ is $o_P(1)$. For the second term on the RHS of (35) note that, due to (27) and (28), $\widehat{\gamma}$ and $\widehat{\beta}$ belong respectively in $\mathcal{N}_\delta(\gamma^0)$ and $\mathcal{N}_\delta(\beta^0)$ with probability approaching one. Hence the second term is $o_P(1)$ by Assumption **A8**(a) and the condition that $h \to 0$. Similar arguments give the third term on the RHS to be $o_P(1)$ by virtue of Assumption **A8**(b). Finally consider the fourth term and note that: $\|M_N^h(\widehat{\beta}_N,\widehat{\gamma}_N,\widetilde{\theta}_N^h)\| \leq \|M_N^h(\widehat{\beta}_N,\widehat{\gamma}_N,\theta^0)\| + o_P(1) = \|M^h(\widehat{\beta}_N,\widehat{\gamma}_N,\theta^0)\| + o_P(1)$ where the first inequality follows from (22) and the second by Assumption **A8**(b). Now, (i) the Lipschitz continuity of M^h in **A8**(a), (ii) continuity of $M(.)$ with respect to β and γ that is implied by Assumptions **A1**(b), **A5**(b) and **A6**(a), along with (iii) (27) and (28) give for $h \to 0$, $\|M^h(\widehat{\beta}_N,\widehat{\gamma}_N,\theta^0)\| = \|M(\widehat{\beta}_N,\widehat{\gamma}_N,\theta^0)\| + o_P(1) = \|M(\beta^0,\gamma^0,\theta^0)\| + o_P(1)$, and this is $o_P(1)$ by (26). Hence the fourth term is also $o_P(1)$ and thus it follows that $\widetilde{\theta}_N^h \xrightarrow[h\to 0]{P} \theta^0$.

Asymptotic equivalence: In a just-identified model, $\widetilde{\theta}_N^h$ satisfies the definition in (22) if

$$o_P(1) = \sqrt{N}M_N^h(\widehat{\beta},\widehat{\gamma}_N,\widetilde{\theta}_N^h).$$

Denoting $\zeta = (\beta', \gamma', \theta')'$ for simplicity, and expanding the RHS we obtain:

$$\begin{aligned} o_P(1) &= \sqrt{N} M_N^h(\zeta^0) + \frac{\partial}{\partial \beta'} M_N^h(\bar{\zeta}_{\beta,N}) \sqrt{N}(\widehat{\beta}_N - \beta^0) + \frac{\partial}{\partial \gamma'} M_N^h(\bar{\zeta}_{\gamma,N}) \sqrt{N}(\widehat{\gamma}_N - \gamma^0) \\ &\quad + \frac{\partial}{\partial \theta'} M_N^h(\bar{\zeta}_{\theta,N}) \sqrt{N}(\widetilde{\theta}_N^h - \theta^0) \end{aligned}$$

for some (row-by-row) mean-values $\bar{\zeta}_{\beta,N}$, $\bar{\zeta}_{\gamma,N}$ and $\bar{\zeta}_{\theta,N}$. Therefore, by $\sqrt{N}$-consistency of $\widehat{\beta}_N$ and $\widehat{\gamma}_N$ from (28) and (27), consistency of $\widetilde{\theta}_N^h$ (just established above), uniform convergence in Assumptions **A8**(c)(i) (applied to the second and third terms on RHS) and **A8**(c)(ii) (applied to the last term on RHS), the continuity assumption in **A8**(d), it follows that

$$o_P(1) = \sqrt{N} M_N^h(\zeta^0) + \frac{\partial M(\zeta^0)}{\partial \zeta'} \sqrt{N} \left[(\widehat{\beta}_N - \beta^0)', (\widehat{\gamma}_N - \gamma^0)', (\widetilde{\theta}_N^h - \theta^0)' \right]'.$$

Finally take $\delta_N > 0$ and $\delta_N = o(N^{-1/2})$, and note that:

$$\begin{aligned} \sup_{h \in (0,\delta_N)} \sqrt{N} \| M_N^h(\zeta^0) - M_N(\zeta^0) \| &\leq \sup_{h \in (0,\delta_N)} \sqrt{N} \| (M_N^h(\zeta^0) - M^h(\zeta^0)) - (M_N(\zeta^0) - M(\zeta^0)) \| \\ &\quad + \sup_{h \in (0,\delta_N)} \sqrt{N} \| M^h(\zeta^0) - M(\zeta^0) \| \\ &\leq o_P(1) + \sqrt{N} b \times \delta_N \end{aligned}$$

with probability approaching 1, respectively by Assumptions **A8** (d) (along with the fact that $M(\zeta^0) = 0$) and (a). Since $\delta_N = o(N^{-1/2})$ as dictated by the statement of the Proposition, it now follows that $\sup_{h \in (0,\delta)} \sqrt{N} \| M_N^h(\zeta^0) - M_N(\zeta^0) \| = o_P(1)$ and hence

$$o_P(1) = \sqrt{N} M_N(\zeta^0) + \frac{\partial M(\zeta^0)}{\partial \zeta'} \sqrt{N} \left[(\widehat{\beta}_N - \beta^0)', (\widehat{\gamma}_N - \gamma^0)', (\widetilde{\theta}_N^h - \theta^0)' \right]' = \sqrt{N} L_N(\widetilde{\theta}_N^h)$$

for $L_N(\theta)$ defined in the proof of **Proposition** 2. Therefore, $\| L_N(\widetilde{\theta}_N^h) \| = o_P(N^{-1/2})$. Now by following the same steps as in that proof we obtain $\sqrt{N} \| \widetilde{\theta}_N^h - \widehat{\theta}_N \| = o_P(1)$. ■

B Appendix B: Further Numerical Evidence

B.1 Model Setup, Missingness Mechanism and Simulation Design

To further study the finite-sample performance of the IPW-II and IPW-GII procedures, we apply these approaches to simulated data from three separate dynamic probit models adapted, respectively, from Models 1, 2 and 3 in Bruins et al. (2016). Recall that the numerical evidence provided in Section 4 of the main text of our paper is adapted from Model 4 in Bruins et al. (2016).

In all three models an individual $i = 1, \ldots, N$ chooses one of the two available alternatives by maximizing the standard additive random utility. In particular, for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, the individuals choice follows $Y_{i,t} = 1(U_{i,t} \geq 0)$, where $U_{i,t}$ is the net utility from the choice of alternative 1 (over alternative 0). The exogenous covariates entering $U_{i,t}$ are $X_{i,t}$ and

are i.i.d. $N(1,2)$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$ across all three models. The particulars of the three models analyzed in this appendix are as follows:

Model 1: $T = 2$ and $U_{i,t} = \lambda X_{i,t} + e_{i,t}$, where $e_{i,t} = \rho e_{i,t-1} + \epsilon_{i,t}$ with $e_{i,0} = 0$ and $\epsilon_{i,t} \sim N(0,1)$. The structural parameters are $\theta = (\lambda, \rho)'$.

Model 2: A slight alternation of Model 1 that includes a lagged dependent variable in the unobserved utility function as follows. $T = 2$ and $U_{i,t} = \alpha Y_{i,t-1} + \lambda X_{i,t} + e_{i,t}$, where $e_{i,t} = \rho e_{i,t-1} + \epsilon_{i,t}$ with $Y_{i,0} = 0$, $e_{i,0} = 0$ and $\epsilon_{i,t} \sim N(0,1)$. The structural parameters are $\theta = (\alpha, \lambda, \rho)'$.

Model 3: Similar to Model 2 but incorporates the well-known 'initial-conditions' problem as follows. Model 2 holds with $T = 5$ but for $i = 1, \ldots, N$, the econometrician observes the choices $Y_{i,t}$ only for $t = 3, 4, 5$. The structural parameters remain $\theta = (\alpha, \lambda, \rho)'$.

Missingness Mechanism: The exogenous covariate $X_{i,t}$ is missing endogenously in these models as follows:

- **Models 1 and 2:** $X_{i,2}$ is only observed when $D_i = 1$, where $D_i = 1(\gamma_1 + \gamma_2 \times Y_{i,1} \geq v_i)$, with $v_i \overset{\text{i.i.d.}}{\sim} N(0,1)$ independent of the structural errors $e_{i,1}, e_{i,2}$ for $i = 1, \ldots, N$. All other variables (except the errors) are always observed for $t = 1, 2$ and $i = 1, \ldots, N$.
- **Model 3:** We confine the missingness of $X_{i,t}$, in the spirit of the other models, to the last time period, i.e., $T = 5$. In particular, $X_{i,5}$ is only observed when $D_i = 1$ where D_i is defined as $D_i = 1(\gamma_1 + \gamma_2 \times Y_{i,4} \geq v_i)$, with $v_i \overset{\text{i.i.d.}}{\sim} N(0,1)$ independent of the structural errors $e_{i,t}$ for $t = 1, \ldots, 5$ and $i = 1, \ldots, N$. All other variables (except the errors) are always observed, except for $Y_{i,1}, Y_{i,2}, Y_{i,3}$ as dictated by Model 3 itself.

The true value of the structural parameters α, λ and ρ, and the missingness-design parameters γ_0 and γ_1 are set to $\alpha^0 = .2$, $\lambda^0 = 1$, $\rho^0 = .4$, $\gamma_1^0 = 0$ and $\gamma_2^0 = -0.5$ respectively.[18]

As a consequence, $X_{i,T}$ is missing in roughly 36% of sample units across Models 1, 2 and 3.

B.2 Auxiliary Model

Our implementation of II in these three models follows the same general steps described in Section 3.1-3.3. For each model we follow Bruins et al. (2016) and consider an auxiliary model based on a linear probability specification for the observed choices. In particular, as in Section 4, we take as the $m(.)$ function a vector of moment restrictions that define the auxiliary parameters β as follows

- For Models 1 and 2 and $i = 1, \ldots, N$

$$m(R_{i,1}, R_{i,2}, X_{i,1}, X_{i,2}; \beta) = \begin{bmatrix} R_{i,1} - \beta_{0,1} - \beta_{1,1} X_{i,1} \\ X_{i,1}(R_{i,1} - \beta_{0,1} - \beta_{1,1} X_{i,1}) \\ R_{i,2} - \beta_{0,2} - \beta_{1,2} X_{i,1} - \beta_{2,2} X_{i,2} - \beta_{3,2} R_{i,1} \\ X_{i,2}(R_{i,2} - \beta_{0,2} - \beta_{1,2} X_{i,1} - \beta_{2,2} X_{i,2} - \beta_{3,2} Y_{i,1}) \end{bmatrix}$$

where $\beta = (\beta_{0,1}, \beta_{1,1}, \beta_{0,2}, \beta_{1,2}, \beta_{2,2}, \beta_{3,2})'$ and R stands for either Y or $Y(\theta)$ as appropriate.

[18] This specification for the structural parameters is the same as in Bruins et al. (2016). The specification of the missingness mechanism is similar to that considered in the Monte Carlo experiments in Section 4.

- For Model 3, we consider exactly the same $m(.)$ function as defined above but with $R_{i,1}, R_{i,2}, X_{i,1}$ and $X_{i,2}$ replaced by $R_{i,4}, R_{i,5}, X_{i,4}$ and $X_{i,5}$ respectively.

Everything else related to the auxiliary model is the same as in Section 4.

B.3 Simulation Results

Following the same pattern of display as in Section 4, we report in Tables 4-6 the mean bias (MBIAS), mean absolute bias (ABIAS), standard deviation (STD), interquartile range (IQR) and the coverage of a 95% Wald-confidence interval (COV95) for the infeasible II, the IPW-II and the IPW-GII estimators based on $S = 10$ (number of simulations) for Models 1-3. All results are based on 10,000 Monte Carlo trials. For brevity, we only report the results for $N = 200$ (small sample) and $N = 5000$ (reasonably large sample); results for additional specifications for N are available from the authors.

Table 3 reports the same Monte Carlo measures for the standard, or 'complete case', II estimator in Models 1-3 with $N = 5000$. Similar to the results in Table 1 of the main text, here also the poor performance in terms of bias and coverage for the standard II estimator is strikingly evident. Not surprisingly, the performance is the worst for the coefficient of the exogenous regressor, whose endogenous missingness is ignored by the standard II estimator.

The results in Tables 4-6 follow the same pattern as Table 2 in the main text. The two feasible estimators perform quite well in all aspects, and are competitive with the infeasible estimator in terms of MBIAS and COV95. Given the missing observations, both these estimators are more variable, as they should be, than the infeasible II estimator, which works with the infeasible data without any missingness in $\{X_{i,t}\}$. Based on these results, albeit under limited but important scenarios, we cannot conclusively rank the relative performance of the IPW-II and IPW-GII estimators. Both seem to work well and as intended in the presence of an endogenously missing exogenous variable.

N=5000	θ	MBIAS	ABIAS	STD	IQR	COV95
Model 1	λ	2.8883	2.8883	0.1838	0.1250	00.00
T=2	ρ	-0.2248	0.2811	0.2687	0.3984	88.82
Model 2	λ	2.9076	2.9076	0.1661	0.1250	00.00
T=2	α	-0.1460	0.2416	0.2707	0.3281	93.37
	ρ	-0.1626	0.2218	0.2336	0.3125	86.05
Model 3	λ	2.8551	2.8551	0.2343	0.1875	00.00
T=5	α	-0.1164	0.2240	0.2639	0.3125	93.94
	ρ	-0.1278	0.2228	0.2582	0.3223	94.48

Table 3: Monte-Carlo results for the standard II estimator in Models 1-3. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, (Monte-Carlo) standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the standard II estimator for the different elements of θ across Models 1-3. Results are based on 10,000 Monte-Carlo trials.

		$N = 200$					$N = 5000$				
Estimator	θ	MBIAS	ABIAS	STD	IQR	COV95	MBIAS	ABIAS	STD	IQR	COV95
	λ	0.0444	0.1262	0.1892	0.1875	94.71	0.0022	0.0229	0.0309	0.0352	93.25
INF-II	ρ	-0.0203	0.3062	0.3827	0.5001	96.38	0.0027	0.0566	0.0712	0.0957	95.44
	λ	0.0481	0.1519	0.2270	0.2031	93.43	0.0019	0.0258	0.0351	0.0386	95.24
IPW-II	ρ	-0.1357	0.4329	0.5226	0.7812	94.62	0.0011	0.0899	0.1131	0.1562	95.35
	λ	0.1340	0.2052	0.3779	0.2725	96.99	0.0054	0.0244	0.0329	0.0365	94.61
IPW-GII	ρ	-0.0148	0.3826	0.5588	0.5484	95.42	-0.0179	0.0478	0.0793	0.0410	91.52

Table 4: Monte-Carlo results for the dynamic probit model in Model 1. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the concerned estimator for the different elements of the parameter vector θ. Results are based on 10,000 Monte-Carlo trials.

		$N = 200$					$N = 5000$				
Estimator	θ	MBIAS	ABIAS	STD	IQR	COV95	MBIAS	ABIAS	STD	IQR	COV95
	λ	0.0466	0.1275	0.2254	0.1875	95.27	0.0021	0.0232	0.0314	0.0352	95.40
INF-II	α	0.0283	0.1699	0.2386	0.2501	91.43	0.0009	0.0362	0.0467	0.0625	94.35
	ρ	-0.0281	0.3104	0.3903	0.5000	95.66	0.0017	0.0575	0.0721	0.0967	95.50
	λ	0.0541	0.1524	0.2865	0.1875	97.11	0.0019	0.0257	0.0351	0.0381	95.13
IPW-II	α	0.0383	0.2453	0.3641	0.3751	93.31	0.0021	0.0550	0.0716	0.0938	96.08
	ρ	-0.1389	0.4319	0.5199	0.7500	94.57	0.0004	0.0915	0.1154	0.1553	95.15
	λ	0.0656	0.1212	0.1601	0.1866	94.11	-0.0054	0.0301	0.0378	0.0506	95.06
IPW-GII	α	0.0325	0.2188	0.2801	0.3575	95.38	-0.0024	0.0390	0.0490	0.0658	95.47
	ρ	-0.0369	0.3427	0.4654	0.5326	95.13	-0.0241	0.0663	0.0792	0.1053	94.41

Table 5: Monte-Carlo results for the dynamic probit model in Model 2. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the concerned estimator for the different elements of the parameter vector θ. Results are based on 10,000 Monte-Carlo trials.

		$N = 200$					$N = 5000$				
Estimator	θ	MBIAS	ABIAS	STD	IQR	COV95	MBIAS	ABIAS	STD	IQR	COV95
INF-II	λ	-0.0166	0.1030	0.1474	0.1562	95.63	-0.0153	0.0283	0.0334	0.0391	93.68
	α	0.0288	0.1664	0.2327	0.2500	91.79	-0.0033	0.0366	0.0468	0.0613	94.34
	ρ	-0.0090	0.2875	0.3601	0.5002	96.25	0.0344	0.0635	0.0718	0.0986	92.75
IPW-II	λ	-0.0284	0.1246	0.1738	0.1562	95.21	-0.0214	0.0336	0.0375	0.0479	92.41
	α	0.0408	0.2399	0.3507	0.3750	93.87	0.0004	0.0544	0.0703	0.0938	96.29
	ρ	-0.1147	0.4169	0.5070	0.8125	95.09	0.0308	0.0929	0.1121	0.1563	94.68
IPW-GII	λ	0.0276	0.1683	0.2768	0.2485	97.14	-0.0198	0.0351	0.0380	0.0509	92.65
	α	-0.0174	0.2921	0.3953	0.4753	96.02	-0.0097	0.0558	0.0693	0.0938	95.20
	ρ	0.0370	0.3469	0.4979	0.5043	95.40	0.0160	0.0856	0.1071	0.1419	95.13

Table 6: Monte-Carlo results for the dynamic probit model in Model 3. MBIAS, ABIAS, STD, IQR and COV95 are the mean bias, absolute bias, standard deviation, interquartile range and coverage of a 95% Wald-type confidence interval for the concerned estimator for the different elements of the parameter vector θ. Results are based on 10,000 Monte-Carlo trials.